

POINCARÉ INEQUALITY AND WELL-POSEDNESS OF THE POISSON PROBLEM ON MANIFOLDS WITH BOUNDARY AND BOUNDED GEOMETRY

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ABSTRACT. Let M be a manifold with boundary and bounded geometry. We assume that M has “finite width,” that is, that the distance $\text{dist}(x, \partial M)$ from any point $x \in M$ to the boundary ∂M is bounded uniformly. Under this assumption, we prove that the Poincaré inequality for vector valued functions holds on M . We also prove a general regularity result for uniformly strongly elliptic equations and systems on general manifolds with boundary and bounded geometry. By combining the Poincaré inequality with the regularity result, we obtain—as in the classical case—that uniformly strongly elliptic equations and systems are well-posed on M in Hadamard’s sense between the usual Sobolev spaces associated to the metric. We also provide variants of these results that apply to suitable mixed Dirichlet–Neumann boundary conditions. We also indicate applications to boundary value problems on singular domains.

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1. INTRODUCTION

Let (M, g) be a smooth m -dimensional Riemannian manifold with smooth boundary and bounded geometry (Definition 3.4). We denote the boundary of M by ∂M , as usual, and we assume that we are given a disjoint union decomposition

$$(1) \quad \partial M = \partial_D M \sqcup \partial_N M,$$

where $\partial_D M$ or $\partial_N M$ are (possibly empty) open subsets of ∂M and \sqcup denotes the disjoint union, as usual. We shall say that the pair $(M, \partial_D M)$ has *finite width* if the distance from any point of M to $\partial_D M$ is *bounded uniformly* on M (Definition 3.6). Our first main results is a Poincaré inequality on M for functions vanishing on $\partial_D M$, under the assumption that the pair $(M, \partial_D M)$ has finite width. We also obtain a Gårding-type inequality that does not require finite width.

Our second main result is, as in the classical case of a bounded domain in a Euclidean space, that the Poisson problem on M with Dirichlet boundary conditions on $\partial_D M$ and Neumann boundary conditions on $\partial_N M$ is well-posed in Hadamard's sense, provided that $(M, \partial_D M)$ has finite width. Our class of manifolds seems to be the largest class of manifolds for which such a direct extension of the classical results holds true. More precisely, let

$$(2) \quad H_D^1(M) := \text{closure}_{H^1} \mathcal{C}_c^\infty(M \setminus \partial_D M),$$

that is, $H_D^1(M)$ is the closure in $H^1(M)$ of the space of smooth functions on M whose support is compact and disjoint from $\partial_D M$. (In all cases that we shall consider, we have $H_D^1(M) = \{u \in H^1(M) \mid u|_{\partial_D M} = 0\}$.) For given $F \in H_D^1(M)^*$, regarded as a distribution on M via the inclusion $\mathcal{C}_c^\infty(M \setminus \partial_D M) \subset H_D^1(M)$, we consider Poisson's problem to *find* $u \in H_D^1(M)$ *such that*

$$(3) \quad \Delta u = F,$$

where the Laplacian is $\Delta := d^*d \geq 0$. This amounts to *homogeneous* Dirichlet boundary conditions on $\partial_D M$ and, if u has at least H^2 regularity, it amounts also to Neumann boundary conditions on $\partial_N M$. We thus show that the Poisson problem (3) is well-posed if $(M, \partial_D M)$ has finite width.

Our third main result is a regularity result for pure Dirichlet boundary conditions on general manifolds with boundary and bounded geometry (thus without the finite width assumption). We then establish the well-posedness in $H^{\ell+1}(M)$ of the Poisson problem (3), provided that F is in the corresponding Sobolev space $H^{\ell-1}(M) \oplus H^{\ell+1/2}(M)$ and that $(M, \partial_D M)$ has finite width. (The case of mixed Dirichlet–Neumann boundary conditions can be treated in the same way, but we choose not to include that here, since it would have greatly increased the size of

the paper.) The reader can recognize here that, in the Euclidean case, this well-posedness result reduces to the classical, fundamental result on the well-posedness of the Poisson–Dirichlet boundary value problem in Sobolev spaces on smooth, bounded domains, [1, 33, 40, 49]. This classical well-posedness result has a very wide scope of applications in mathematics and beyond. See also [11] for some recent results. Our results seem to encompass the largest class of manifolds for which such a direct extension of the classical results on the Poisson–Dirichlet problem holds true. This is because, first, one needs bounded geometry to have some well-behaved Sobolev spaces [3, 9, 37, 34]. Second, let Ω in \mathbb{R}^N be a domain that, outside a compact set, coincides with a cone. For this domain, the Poincaré inequality fails. Hence, also the well-posedness of the Poisson–Dirichet problem in $H^1(\Omega)$ fails. This shows that the assumption that we have finite width is necessary in general. For the Poincaré inequality and, hence, also for well-posedness in H^1 , one can however relax some assumptions on the smoothness of the boundary: Lipschitz regularity suffices in this case. This is discussed briefly in Subsection 2.6.

Our well-posedness result is of interest in itself—as a general result on analysis on non-compact manifolds—but also because it has applications to partial differential equations on singular and non-compact spaces, which are more and more often studied. Examples are provided by analysis on curved space-times in general relativity; see for instance the recent solution of Klainerman, Rodnianski, and Szeftel of the bounded curvature condition and the references therein [43]. Other examples are provided by [42, 55, 62]. Manifolds of bounded geometry and their analysis play a role also in Quantum Field Theory on curved space-time [12, 13, 23, 32]. Asymptotically hyperbolic manifolds are important examples of manifolds with bounded geometry, and they play an increasing role in geometry and physics [5, 18, 26, 35]. It was shown in [51] that every Riemannian manifold is conformal to a manifold with bounded geometry. Moreover, manifolds of bounded geometry can be used to study boundary value problems on singular domains, see, for instance, [19, 24, 25, 46, 52] and the references therein. As a first, straightforward example of an application of our Poincaré inequality to singular domains, we recover some results of Kondratiev [44] and Mazya [46], see below Corollary 5.16. One of the main reasons why singular spaces and domains are interesting is that most of the physical domains that arise in practice are not smooth.

A large part of the paper is devoted to proving the Poincaré inequality in our setting. Among the many possible approaches to the proof of this result, we chose one that is based on the normal exponential map. This required us to prove some estimates on the volume distortion of the normal exponential map using a special case of the Heintze–Karcher inequality [10, 41]. This leads to additional insight on the analysis of manifolds with bounded geometry and may turn out to be useful to obtain Hardy type inequalities, see for instance the paper by M. and T. Hoffmann-Ostenhof, Laptev, and Tidblom [39]. See also [4, 47]. In order to apply our result also to uniformly strongly elliptic systems of PDE’s or, more generally, to uniformly strongly elliptic operators acting on sections of vector bundles, such as the Dirac operator, we formulate everything directly for such operators. In particular, we prove our Poincaré inequality for vector valued functions. To this end, we will need a further assumption on the coefficient vector bundle E , namely E will be assumed to be also of bounded geometry (see Definition 3.8).

The paper is organized as follows. Section 2 is devoted to preliminaries on Sobolev spaces on manifolds with boundary, on strongly elliptic operators, and on the relations between these operators, the Lax–Milgram lemma, and the Poincaré inequality. We also discuss in Subsection 2.6 extensions of the Poincaré inequality that allow to treat also Lipschitz boundaries. Section 3 is devoted to some geometric background material, to a statement of our Poincaré inequality, and to some preliminary results on this inequality. The proof of the Poincaré inequality is completed in Section 4, where we also prove, as an application, that the Poisson problem on M , (3), is well-posed in $H_D^1(M; E)$, provided that $(M, \partial_D M)$ has finite width. In Section 5, we recall the definition of Sobolev spaces on manifolds with bounded geometry using partitions of unity and use it to obtain higher regularity for Dirichlet boundary conditions. Then we use the higher regularity to prove the well-posedness of the Poisson problem in $H^{\ell+1} \cap H_0^1$ if the data is in $H^{\ell-1}$, $\ell \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We include also a short discussion of extensions of our result. The first extension is to mixed Dirichlet–Neumann boundary conditions. The second extension is to mixed boundary value problems on domains with conical points in two dimensions, in which case we recover some results of Kondratiev [44] and Mazya [46]. These extensions are one of the main motivations of this paper and will be treated in greater generality somewhere else. We conclude our paper with a characterization of manifolds with boundary and bounded geometry, which, while important, can be skipped by the reader interested only in analysis applications.

2. THE POINCARÉ INEQUALITY: MOTIVATION AND BASIC RESULTS

In this section, we set up some notation and then we discuss the relation between the Poincaré inequality and a suitable form of the well-posedness of the Poisson–Dirichlet problem. This is in preparation for the applications to well-posedness of the Poisson problem on manifolds with finite width (Definition 3.6). All results of this section are quite standard (and well known), although the setting and presentation (in the framework of smooth manifolds) are less common. We begin this section by assuming only that M is a Riemannian manifold with (rough) boundary $\partial M \subset M$, but soon we impose conditions on the curvature R^M of (the Levi–Civita connection of) M . We do *not* require positive injectivity radius in this section. Also, for simplicity, we only consider differential operators with bounded coefficients.

2.1. Notation: vector bundles and operators. We will introduce Sobolev spaces for functions with values in a vector bundle E . In this paragraph we will recall and introduce some notation for vector bundles mostly to fix some notation.

For complex vector spaces V and W , a *sesquilinear* map $V \times W \rightarrow \mathbb{C}$ will always be (complex) linear in V and antilinear in W . A *Hermitian form* on V is a positive definite sesquilinear map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$, which implies $\langle w, v \rangle = \overline{\langle v, w \rangle}$. In order to deal with the complex version of the Lax–Milgram lemma (which is much less used than the real version, we introduce the following notation.

Notations 2.1. *Let V be a complex vector space, usually a Hilbert space. We shall denote by \overline{V} the complex conjugate vector space to V , it is V as an additive group, but with the complex structure $\lambda J_0(v) = J_0(\overline{\lambda}v)$, where $J_0 : V \rightarrow \overline{V}$ is the canonical identification and $\overline{\lambda}$ is the complex conjugate of $\lambda \in \mathbb{C}$. If V is endowed with a topology, we denote by V' the (topological) dual of V . It will be convenient to denote $V^* := (\overline{V})'$. We can thus regard a continuous sesquilinear form $B : V \times W \rightarrow \mathbb{C}$*

as a continuous bilinear form on $V \times \overline{W}$, or, moreover, as a map $V \rightarrow W^*$. Let us assume V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$. We shall then extend inner product to a map $V^* \times V \rightarrow \mathbb{C}$ denoted in the same way: $F(v) = \langle F, v \rangle$.

Let us assume that $E \rightarrow M$ is a smooth complex vector bundle endowed with a Hermitian form a metric preserving connection $\nabla^E : \Gamma(M; E) \rightarrow \Gamma(M; E \otimes T^*M)$. We consider the tangent bundle $TM \rightarrow M$ as a (real) vector bundle with the (real) inner product given by the Riemannian metric and the Levi-Civita connection which is product preserving connection ∇^M . If \otimes denotes the real tensor product, then $E \otimes T^*M^{\otimes j}$ is a vector bundle, where the complex structure of E defines is used to define complex multiplication on $E \otimes T^*M^{\otimes j}$, so it is again a complex vector bundle. The product metric of the hermitian metric on E and the standard inner products on T^*M define a hermitian metric on $E \otimes T^*M^{\otimes j}$, and it is preserved by the product connection of ∇^E and ∇^M . Often we write simply ∇ for ∇^E , for ∇^M and for the product connection, when there is no danger of confusion. Similar notation and structures will be used when tensoring with TM instead of T^*M . As in 2.1, the (complex) conjugated bundle \overline{E} is obtained by taking the structure of E as real vector bundle, but replacing the complex multiplication by i with multiplication with $-i$. For complex vector bundles E and F , we write $\text{Hom}(E, F)$ for the complex homomorphisms, and $\text{End}(E) := \text{Hom}(E, E)$. The trivial complex vector bundle of rank k with the trivial connection is written as $\underline{\mathbb{C}}^k \rightarrow M$ or simply as $\underline{\mathbb{C}}^k$. The (complex) *dual bundle* is defined as $E' := \text{Hom}(E, \underline{\mathbb{C}})$, and we define the *conjugated dual bundle* as $E^* := \overline{E'}$, so the elements of E^* are fiberwise anti-linear homomorphisms from E to $\underline{\mathbb{C}}$. Thus the Hermitian form can be used to identify E with E^* which we will usually do.

Let X and Y be Banach spaces and $A : X \rightarrow Y$ be a linear map. Recall that A is *continuous* if, and only if, $\|A\|_{X,Y} := \inf_{x \neq 0} \frac{\|Ax\|}{\|x\|} < \infty$. (An operator with the later property is called *bounded*, so we will sometimes say “bounded operator” instead of “continuous operator.”) When the spaces on which A acts are clear, we shall drop them from the notation of the norm. We say that A is an *isomorphism* if it is a continuous bijection, (in which case the inverse will also be continuous, by the open mapping theorem). If $T : V \rightarrow W$ is a continuous map of *Hilbert* spaces, then we denote by $T^* : V^* \rightarrow W^* := (\overline{W'})' \cong (\overline{W'})$ the *adjoint* of T , as usual. We note that it is \mathbb{C} -linear, since a map $A : X \rightarrow Y$ is \mathbb{C} -linear if, and only if $A : \overline{X} \rightarrow \overline{Y}$ is \mathbb{C} -linear.

2.2. Sobolev spaces and differential operators. In this subsection, we recall the definitions of the needed Sobolev spaces and we introduce the differential operators that we will consider in this paper.

Throughout this paper, M will be a (usually non-compact) connected smooth Riemannian manifold of dimension $m+1$ with boundary ∂M and metric g and volume form dvol_g . While, in general, M will be as smooth as possible, the boundary will be allowed, occasionally, to have lower regularity.

We shall assume throughout the paper that we are given a decomposition $\partial M = \partial_D M \sqcup \partial_N M$ with $\partial_D M$ and $\partial_N M$ open in ∂M , as in Equation (1). In particular, each of $\partial_D M$ and $\partial_N M$ is also closed and a disjoint union of connected components of ∂M . In our applications to boundary value problems, the set $\partial_D M$ is the set of boundary points where *Dirichlet* boundary conditions are imposed, whereas $\partial_N M$ is the set of boundary points where *Neumann* boundary conditions are imposed.

We also fix a complex vector bundle $E \rightarrow M$ with a (fiberwise) Hermitian form $\langle \cdot, \cdot \rangle_E$ and a product preserving connection ∇^E . As usual, $\Gamma_c(M; E) \subset \Gamma(M; E) \subset \Gamma_{me}(M; E)$ denote the spaces of *smooth, compactly supported* sections of E , respectively *smooth*, respectively, *(equivalence classes of) measurable* sections of E . The L^p -norm $\|u\|_{L^p(M; E)}$ of a section $u \in \Gamma(M; E)$ is then

$$\|u\|_{L^p(M; E)}^p := \int_M |u(x)|^p \, d\text{vol}_g(x), \quad \text{if } 1 \leq p < \infty, \text{ and}$$

$$\|u\|_{L^\infty(M; E)} := \text{ess-sup}_{x \in M} |u(x)|,$$

as usual. (We set $\|u\|_{L^p(M; E)} := \infty$ if u is not p -summable.)

We now introduce the first definition of Sobolev spaces. See [8, 9, 30, 28, 36, 38, 61, 64, 68] for history, further results, and further references. We consider them on manifolds and bundles of bounded geometry, see [3, 2, 6, 23, 34, 45, 65, 60] for results in this setting and Section 3 for the definition of bounded geometry.

Let $\ell \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We define

$$W^{\ell, p}(M; E) := \{u \in \Gamma_{me}(M; E) \mid \nabla^j u \in L^p(M; E \otimes T^* M^{\otimes j}), \, j \leq \ell\}$$

$$\|u\|_{W^{\ell, p}(M; E)}^p := \sum_{k=0}^{\ell} \|\nabla^k u\|_{L^p(M; E \otimes T^* M^{\otimes k})}^p, \quad \text{for } p < \infty,$$

$$\|u\|_{W^{\ell, \infty}(M; E)} := \max_{k=0}^{\ell} \|\nabla^k u\|_{L^p(M; E \otimes T^* M^{\otimes k})}.$$

We let $W^{\infty, p} := \bigcup_{\ell} W^{\ell, p}$. When $E = \underline{\mathbb{C}} := M \times \mathbb{C}$, the trivial one-dimensional bundle, we omit it from the notation. Thus $W^{\ell, p}(M) := W^{\ell, p}(M; \underline{\mathbb{C}})$. For the fiber over $x \in M$, we write E_x . Recall that write V' for the *topological dual* of a vector space V . The pointwise trace $\text{tr}_x: E_x \otimes_{\mathbb{R}} E'_x \rightarrow \mathbb{C}$, $x \in M$, defines the canonical contraction $E \otimes_{\mathbb{R}} E' \rightarrow \underline{\mathbb{C}}$ and thus a continuous map

$$(4) \quad W^{\ell, p}(M; E \otimes_{\mathbb{R}} E') \rightarrow W^{\ell, p}(M).$$

For more general maps of this form, we will need to consider products. Indeed, the (complex) tensor product defines a continuous map

$$(5) \quad W^{\ell, \infty}(M; E) \times W^{\ell, p}(M; E_1) \ni (u, v) \mapsto u \otimes v \in W^{\ell, p}(M; E \otimes_{\mathbb{R}} E_1),$$

whereas evaluation in E defines a continuous map

$$W^{\ell, \infty}(M; \text{Hom}(E; E_1)) \times W^{\ell, p}(M; E) \ni (a, u) \mapsto au \in W^{\ell, p}(M; E_1).$$

We now introduce the concepts of “manifold with totally bounded curvature” and “manifold with bounded geometry,” which will be needed to formulate the assumptions of most of the results in this article. We shall say that a (real or complex) vector bundle E with given connection *has totally bounded curvature* if its curvature is bounded and if so are all its covariant derivatives. In the literature, a Riemannian manifold M without boundary is said to be of *bounded geometry* if TM has totally bounded curvature and if M has positive injectivity radius. As we are interested in manifolds M with boundary, we will discuss extensions of this concept to manifolds with boundary in Section 3, see also Section 6, following mainly [58]. Thus, we say that E *is of bounded geometry*, if E has totally bounded curvature and if M is of bounded geometry, see Definition 3.8 for details.

We shall assume from now on that $E \rightarrow M$ and $TM \rightarrow M$ have totally bounded curvature. Let us now introduce a class of compatible differential operators. Let

$a_j \in W^{\ell,\infty}(M; \text{End}(E) \otimes TM^{\otimes j})$, $0 \leq j \leq k$. In particular, $a_0 \in W^{\ell,\infty}(M; \text{End}(E))$. Note that one needs to consider vector valued Sobolev spaces even if one is interested only in scalar equations. Then we say that

$$(6) \quad Pu := \sum_{j=0}^k a_j \nabla^j u,$$

is a *differential operator of order k with coefficients in $W^{\ell,\infty}$* , where the products are obtained from

$$W^{\ell,\infty}(M; \text{End}(E) \otimes TM^{\otimes j}) \otimes W^{\ell,p}(M; T^*M^{\otimes j} \otimes_{\mathbb{R}} E) \rightarrow W^{\ell,p}(M; E)$$

by the contraction of tensors, using the isomorphism $T^*M^{\otimes j} \simeq (TM^{\otimes j})^*$, which is induced by the Riemannian metric on M . We denote this differential operator by $P = \sum_{j=0}^k a_j \nabla^j$, for short. It defines a *continuous map*

$$(7) \quad P = \sum_{j=0}^k a_j \nabla^j : W^{\ell+k,p}(M; E) \rightarrow W^{\ell,p}(M; E).$$

If $\ell = 0$, we shall say that P has *bounded* coefficients. If $\ell = \infty$, we shall say that P has *totally bounded* coefficients.

Let us define

$$(8) \quad W_D^{\ell,p}(M; E) := \text{closure}_{W^{\ell,p}(M; E)} \Gamma_c(M \setminus \partial_D M; E),$$

the closure of $\Gamma_c(M \setminus \partial_D M; E)$ in $W^{\ell,p}(M; E)$. Recall that $\Gamma_c(M \setminus \partial_D M; E)$ is the space of smooth sections of $E \rightarrow M$ that have compact support not intersecting $\partial_D M$.

2.3. Strongly elliptic operators. Our main interest lies in the case $p = 2$, because the Hilbert space structure of $W^{\ell,2}(M; E)$ is helpful. In this case, we use the special notations:

$$(9) \quad H^{\ell}(M; E) := W^{\ell,2}(M; E), \quad H_D^{\ell}(M; E) := W_D^{\ell,2}(M; E).$$

The sesquilinear Hilbert space product is written as

$$(u, v) := \int_M \langle u, v \rangle_E \, \text{dvol}_g, \quad u, v \in H^{\ell}(M; E).$$

We shall need also *negative order* Sobolev spaces. One approach is to use duality. If M has no boundary and $s \in \mathbb{R}$, then the spaces $H^s(M; E)$ are defined by interpolating the spaces $H^{\ell}(M; E)$, $\ell \in \mathbb{Z}$. See [49, 63] for the case of manifolds with boundary. We again set $\|u\|_{H^s(M; E)} = \infty$ for $u \notin H^s(M; E)$, using the convention that $C \times \infty = \infty$ for $C \in (0, \infty]$.

If the boundary is not empty but $\partial_N M = \emptyset$ (that is, if $\partial_D M = \partial M$), we shall write $H_0^{\ell} = H_D^{\ell}$ and $W_0^{\ell,p} = W_D^{\ell,p}$, as usual. Furthermore, we define $H^{-\ell}(M; E)$ as the dual of $H_0^{\ell}(M; E')$, where $E' = \text{Hom}(E, \mathbb{C})$ is the dual bundle of E . For $u \in H^{-\ell}(M; E)$, the composition $\Gamma_c(M; E') \hookrightarrow H_0^{\ell}(M; E') \xrightarrow{u} \mathbb{C}$ allows us to view u as a distributional section of E , and hence $H^{\ell}(M, E) \subset L^2(M, E) \subset H^{-\ell}(M, E)$. As in 2.1, we shall always use the Hermitian form to identify E' with \overline{E} , which leads to $H_0^{\ell}(M; E') \cong H_0^{\ell}(M; \overline{E}) \cong \overline{H_0^{\ell}(M; E)}$. Recalling that $V^* := (\overline{V})' \cong \overline{V'}$, see 2.1, this finally gives in the hermitian case

$$(10) \quad H^{-\ell}(M, E) := H_0^{\ell}(M; E)^* := \overline{H_0^{\ell}(M; E)}' \cong H_0^{\ell}(M; \overline{E})'.$$

Furthermore, the (complex bilinear) duality pairing $H^{-\ell}(M; E) \times H_0^\ell(M; \overline{E}) \rightarrow \mathbb{C}$ can be viewed as a sesquilinear map $H^{-\ell}(M; E) \times H_0^\ell(M; E) \rightarrow \mathbb{C}$. This map coincides with the hermitian L^2 -product $L^2(M; E) \times L^2(M; E) \rightarrow \mathbb{C}$ on the intersection of their domains, so in both pairings we can write simply $\langle u, v \rangle$ for the product of u and v . This discussion and its consequences are often summarized by saying that $H^{-\ell}(M; E)$ is *the (conjugate) dual of $H_0^\ell(M; E)$ with pivot $L^2(M, g)$* . If $\partial_N \neq \emptyset$, we shall proceed by analogy and consider the spaces $H_D^\ell(M; E)^* \cong H_D^\ell(M; \overline{E})'$. We note that in many applications E is the complexification of a real vector bundle, and hence there is a canonical isomorphism $E \cong \overline{E}$, which explains why the complex conjugate is often omitted.

It is important in applications to consider operators “in divergence form,” which we will define shortly below after providing a few more preliminaries. The operators in divergence form provide a slightly different class of differential operators than the operators with coefficients in L^∞ considered above. We restrict ourselves, for simplicity, to second order operators and $p = 2$. For any Banach spaces V and W , we have a natural bijective correspondence between the space of continuous, linear maps $P: V \rightarrow W^* := \overline{W}'$ and the space of continuous, sesquilinear forms $B: V \times W \rightarrow \mathbb{C}$ given by $B(v, w) := P(v)(w) = \langle P(v), w \rangle$. We shall use this observation for $V = W = H_D^1(M; E)$ as follows. Let us assume that, for each $x \in M$, we have a sesquilinear map $a_x: T_x^*M \otimes_{\mathbb{R}} E_x \times T_x^*M \otimes_{\mathbb{R}} E_x \rightarrow \mathbb{C}$. This leads to a section a of the bundle $((T^*M \otimes_{\mathbb{R}} E) \otimes_{\mathbb{C}} (T^*M \otimes_{\mathbb{R}} \overline{E}))'$. We say that $a = (a_x)_{x \in M}$ is a *bounded family of sesquilinear forms on $T_x^*M \otimes_{\mathbb{R}} E_x$* if all a_x are sesquilinear as above and a is an L^∞ -section (including measurability). The *Dirichlet form B_a on $H_D^1(M; E)$* associated to a is

$$(11) \quad B_a(u, v) = B_a^g(u, v) := \int_M a(\nabla u, \nabla v) \, \text{dvol}_g.$$

In particular

$$B_a: H_D^1(M; E) \times H_D^1(M; E) \rightarrow \mathbb{C}$$

is sesquilinear, and thus gives an operator

$$P_a^g: H_D^1(M; E) \longrightarrow H_D^1(M; E)^* \cong H_D^1(M; \overline{E})'.$$

In other words, P_a^g is the uniquely determined linear operator such that $P_a^g(v)(w) = \langle P_a^g(v), w \rangle = B_a^g(v, w)$ for all $v \in H_D^1(M; E)$ and $w \in H_D^1(M; E)$. Since B_a^g depends on the choice of the metric g , so does P_a^g , although this will typically *not* be shown in the notation, since most of the time the metric will be fixed.

Example 2.2. Let a be the hermitian product on $T^*M \times E$ obtained as the product of the Riemannian metric on T^*M and the Hermitian form on E . Then

$$\nabla^* \nabla := \text{res} \circ P_a: H_D^1(M; E) \rightarrow H^{-1}(M; E)$$

is the connection Laplacian which is locally given by the formula

$$\nabla^* \nabla u = \sum_{j=1}^n (-\nabla_{e_j} \nabla_{e_j} u + \nabla_{\nabla_{e_j} e_j} u),$$

where (e_1, \dots, e_n) is a local orthonormal frame. In the special case $E = \mathbb{C}$ with the trivial connection and Hermitian form, this operator is called the Laplace-Beltrami operator and we write Δ_g for $\nabla^* \nabla$ in this case. Often in the literature, Δ_g is called the geometer’s (or positive) Laplacian, in order to fix the sign convention.

We shall use the fact that the map 'res' defined by restriction maps elements in $H_D^1(M; E)^*$ to elements in $H^{-1}(M; E) := H_0^1(M; E)^*$, see Equation (10). Then, for general regular enough a , the operator $\text{res} \circ P_a$ will be a differential operator $a_2 \nabla^2 + a_1 \nabla$ in the classical sense, where $a_k \nabla^k$ denotes the composition $\nabla^k: \Gamma(E) \rightarrow \Gamma(T^*M^{\otimes k} \otimes_{\mathbb{R}} E)$ with some coefficient $a_k: \Gamma(T^*M^{\otimes k} \otimes_{\mathbb{R}} E) \rightarrow \Gamma(E)$. Using the hermitian metric to map $L^2(M; E)$ to $H_0^1(M; E)^*$, an integration by parts argument shows that a_2 corresponds to $-a$.

We are interested in operators $H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ because we have the equivalence of the following formulations (if the boundary is nice enough, see Remark 2.3).

- (i) The operator $P: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism.
- (ii) The boundary value problem

$$(12) \quad \begin{cases} Pu = F \in H_D^1(M; E)^* & \text{in } M \\ u = h \in H^{1/2}(\partial_D M; E|_{\partial_D M}) & \text{on } \partial_D M \end{cases}$$

has a unique solution $u \in H^1(M; E)$, depending continuously on F and h .

- (iii) For each $F \in H_D^1(M; E)^*$ and $h \in H^{1/2}(\partial_D M; E|_{\partial_D M})$, the "weak" problem

$$(13) \quad \begin{cases} \langle P(u), v \rangle = \langle F, v \rangle & \text{for all } v \in H_D^1(M; E) \\ u = h & \text{on } \partial_D M \end{cases}$$

has a unique solution $u \in H^1(M; E)$, depending continuously on F and h .

Remark 2.3. Let us assume that

$$(14) \quad H_D^1(M; E) = \{ u \in H^1(M; E) \mid u|_{\partial_D M} = 0 \},$$

see also Equations (2), (8), and (9).

If M is a compact manifold with smooth boundary, then this assumption (Equation (14)) is a classical result. In our case of manifolds with boundary and bounded geometry, where also the coefficient bundle has totally bounded curvature, it is a consequence of the results in [34] (see Theorem 5.8). The equivalence of the above problems relies, of course, on the surjectivity of the map $H^\ell(M; E) \rightarrow H^{\ell-1/2}(\partial M; E)$ and on the assumption (14). This reconciles the formulations of our results as isomorphisms of certain spaces with the more classical formulations of boundary value problems. Nevertheless, we shall often explicitly provide also the classical formulations of our results.

In order to study the invertibility of operators P as in the formulation (i), one often requires "coercivity." Recall the following concept (we use the terminology of [63]).

Definition 2.4. A bounded family $a = (a_x)_{x \in M}$ of sesquilinear forms on $T^*M \otimes_{\mathbb{R}} E$ will be called *coercive* if there exists $c_a > 0$ satisfying

$$(15) \quad \Re a(\xi, \xi) \geq c_a \|\xi\|^2, \quad \text{for all } \xi \in T^*M \otimes_{\mathbb{R}} E.$$

Note that this definition depends on the choice of the metric g and the Hermitian form on E , which is, however, not shown in the notation. We can now introduce the operators in which we are interested.

Definition 2.5. Let P be a second order (linear) differential operator on $E \rightarrow M$. We shall say that P is a *second order operator in divergence form* if there exist

a bounded family $a = (a_x)_{x \in M}$ of sesquilinear forms on $T^*M \otimes_{\mathbb{R}} E$ and a first order differential operator $Q: H_D^1(M; E) \rightarrow L^2(M; E)$ with coefficients in L^∞ such that $P := P_a + Q: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ continuously. Here we used that $L^2(M, E) \cong L^2(M, E)^*$ embeds into $H_D^1(M; E)^*$. If, moreover, a is coercive, then we shall say that P is *uniformly strongly elliptic*.

One could add to P as in the definition above also a term of the form Q_1^* , where Q_1 is of the same form as Q , but we shall not need this generality here.

If $\partial_D M = \partial M$, the continuous map $P: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ defined by P becomes the usual map $P: H_0^1(M; E) \rightarrow H^{-1}(M; E) := H_0^1(M; E)^*$. We shall also need the following well-known Gårding inequality. We define the semi-norm $|u|_{H^1(M; E)} := \|\nabla u\|_{L^2(M; T^*M \otimes_{\mathbb{R}} E)}$. Obviously $\|u\|_{H^1(M; E)}^2 = |u|_{H^1(M; E)}^2 + \|u\|_{L^2(M; E)}^2$.

Lemma 2.6 (Gårding). *Assume that $Pu := P_a u + Qu$ is uniformly strongly elliptic on $E \rightarrow M$, where $\Re a(\xi, \xi) \geq c_a \|\xi\|^2$. If $\eta < c_a$, then there is $R_\eta \in \mathbb{R}$ such that*

$$\Re \langle Pu, u \rangle \geq \eta |u|_{H^1(M; E)}^2 - R_\eta \|u\|_{L^2(M; E)}^2, \text{ for any } u \in H_D^1(M; E).$$

For brevity we often omit $(M; E)$ in the notation for the (semi-)norms. For instance, $|u|_{H^1} := |u|_{H^1(M; E)}$.

Proof. We include the standard proof, for the convenience of the reader. The operator P is associated to the sesquilinear map

$$(16) \quad \langle Pu, v \rangle := B_a(u, v) + \langle Qu, v \rangle.$$

The operator Q defines a continuous map $H_D^1(M; E) \rightarrow L^2(M; E)$, and we write $\|Q\|_{H^1, L^2}$ for the associated operator norm. We have

$$(17) \quad \begin{aligned} \Re \langle Pu, u \rangle &= \Re B_a(u, u) + \Re \langle Qu, u \rangle = \int_M \Re a(\nabla u, \nabla u) \, \text{dvol}_g + \Re \langle Qu, u \rangle \\ &\geq c_a |u|_{H^1}^2 + \Re \langle Qu, u \rangle. \end{aligned}$$

The inequality $|\langle Qu, u \rangle| \leq \|Q\|_{H^1, L^2} \|u\|_{H^1} \|u\|_{L^2} \leq \epsilon \|u\|_{H^1}^2 + \frac{\|Q\|_{H^1, L^2}^2}{4\epsilon} \|u\|_{L^2}^2$, valid for any $\epsilon > 0$, then gives

$$\Re \langle Pu, u \rangle \geq c_a |u|_{H^1}^2 - \epsilon \|u\|_{H^1}^2 - \frac{\|Q\|_{H^1, L^2}^2}{4\epsilon} \|u\|_{L^2}^2 = (c_a - \epsilon) |u|_{H^1}^2 - R_\eta \|u\|_{L^2}^2,$$

for $R_\eta = \frac{\|Q\|_{H^1, L^2}^2}{4\epsilon} + \epsilon$, for $\epsilon := c_a - \eta$. This proves the lemma. \square

Note that Lemma 2.6 immediately gives the following more standard form of Gårding's inequality

$$(18) \quad \Re \langle Pu, u \rangle \geq \eta \|u\|_{H^1}^2 - (\eta + R_\eta) \|u\|_{L^2}^2, \text{ for any } u \in H_D^1(M; E).$$

2.4. The Lax–Milgram lemma and well-posedness in H^1 . Recall that M is a connected smooth Riemannian manifold of dimension m with boundary $\partial M = \partial_D M \sqcup \partial_N M \subset M$ and metric g . The boundary is not required to be smooth, but we continue to assume, as throughout this section, that Assumption (14) is satisfied and that TM and the coefficient vector bundle $E \rightarrow M$ have totally bounded curvatures.

We shall need the Lax–Milgram lemma, which we recall next. See, for example, Section 5.8 of [33]. Let V be a Hilbert space and let $P: V \rightarrow V^*$ be a bounded

operator. We say that P is *coercive* if there exists $\gamma > 0$ such that $\Re\langle Pu, u \rangle \geq \gamma \|u\|_V^2$.

Lemma 2.7 (Lax–Milgram lemma). *Let V be a Hilbert spaces. Let $P: V \rightarrow V^*$ be a coercive map with $\Re\langle Pu, u \rangle \geq \gamma \|u\|_V^2$. Then P is invertible and $\|P^{-1}\| \leq \gamma^{-1}$.*

Combining with the Gårding's inequality, we immediately obtain the following.

Corollary 2.8. *We use the notation in Lemma 2.6 above. If $c \in L^\infty(M, \text{End}(E))$ satisfies $\Re(c) \geq R_\eta + \epsilon$, for some $\eta, \epsilon > 0$, then $P + c: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism, where η and R_η are as in Lemma 2.6.*

Note that $\Re(c) \geq R_\eta$ means $\Re\langle cu, u \rangle \geq R_\eta \|u\|_{L^2}^2$ for all $u \in L^2(M; E)$.

Proof. We use Lemma 2.6 and its notation (including its proof). In particular, we have $\Re\langle Pu, u \rangle \geq \eta \|u\|_{H^1}^2 - R_\eta \|u\|_{L^2}^2$ for any $u \in H_D^1$. If $\Re(c) \geq R_\eta + \epsilon$, we obtain that $P_1 := P + c$ satisfies $\Re\langle P_1 u, u \rangle \geq \eta \|u\|_{H^1}^2 + \epsilon \|u\|_{L^2}^2 \geq \eta' \|u\|_{H^1}^2$, where $\eta' := \min\{\eta, \epsilon\}$. Therefore P_1 satisfies the assumptions of the Lax–Milgram lemma and hence it is invertible, as claimed. \square

We obtain the following regularity corollary.

Corollary 2.9. *Let P be a uniformly strongly elliptic differential operator on $E \rightarrow M$. Then there exists $\mu > 0$ such that*

$$\|u\|_{H^1(M; E)} \leq \mu (\|Pu\|_{H_D^1(M; E)^*} + \|u\|_{H_D^1(M; E)^*}).$$

Since $L^2(M; E)$ maps continuously to $H_D^1(M; \overline{E})' \cong H_D^1(M; E)^*$, we also obtain that

$$\|u\|_{H^1(M; E)} \leq \mu (\|Pu\|_{H_D^1(M; E)^*} + \|u\|_{L^2(M; E)}).$$

Proof. Using Corollary 2.8, we can find a constant $c > 0$ such that

$$P + c: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$$

is an isomorphism. Then there exists $\gamma > 0$ such that

$$\|u\|_{H^1(M; E)} \leq \gamma \|(P + c)u\|_{H_D^1(M; E)^*}.$$

The triangle inequality then gives the result for $\mu := \max\{\gamma, \gamma c\}$. \square

2.5. The Poincaré inequality and well-posedness in H^1 . In this subsection, we discuss the relation between the Poincaré inequality and well-posedness in H^1 for the Poisson problem with suitable mixed boundary conditions as well as for other uniformly strongly elliptic problems with mixed boundary conditions. This is needed for one of the main results of this paper, which is the well-posedness of the Poisson problem with suitable mixed boundary conditions on manifolds with boundary and finite width (see Definition 3.6 and Theorem 4.8). We continue to assume, as throughout the rest of the paper, that M is as explained in the beginning of Subsection 2.4.

We define the semi-norm $|u|_{W^{1,p}(M; E)} := \|\nabla u\|_{L^p(M; T^*M \otimes_{\mathbb{R}} E)}$.

Definition 2.10. We say that the pair $(E \rightarrow M, \partial_D M)$ satisfies the *p-Poincaré inequality* if there exists $c_p > 0$ such that

$$\|u\|_{L^p(M; E)} \leq c_p \|\nabla u\|_{L^p(M; T^*M \otimes_{\mathbb{R}} E)} =: c_p |u|_{W^{1,p}(M; E)},$$

for all $u \in W_D^{1,p}(M; E)$.

We have the following standard lemma.

Lemma 2.11. *The p -Poincaré inequality is satisfied for $(E \rightarrow M, \partial_D M)$ if, and only if, the semi-norm $|\cdot|_{W^{1,p}(M;E)}$ is equivalent to the $W^{1,p}$ -norm on $W_D^{1,p}(M;E)$.*

Proof. For the simplicity of the notation, we omit below the manifold M and the coefficient vector bundle $E \rightarrow M$ from the notation of the (semi-)norms. Let us assume that $(E \rightarrow M, \partial_D M)$ satisfies the p -Poincaré inequality. We have $|u|_{W^{1,p}} \leq \|u\|_{W^{1,p}}$, so to prove the equivalence of the norms it is enough to show that there exists $C > 0$ such that $C|u|_{W^{1,p}} \geq \|u\|_{W^{1,p}}$ for all $u \in W_D^{1,p}$. Indeed, for $p < \infty$ the p -Poincaré inequality gives

$$(c_p^p + 1)|u|_{W^{1,p}}^p \geq \|u\|_{L^p}^p + |u|_{W^{1,p}}^p =: \|u\|_{W^{1,p}}^p.$$

Conversely, if the two norms are equivalent, then we have for $u \in W_D^{1,p}$

$$\|u\|_{L^p} \leq \|u\|_{W^{1,p}} \leq C|u|_{W^{1,p}} := C\|\nabla u\|_{L^p}.$$

The proof is now complete for $p < \infty$. The case $p = \infty$ is completely similar. \square

Recall from Subsection 2.1 that Q^* denotes the adjoint of a continuous linear operator Q . The following result is standard.

Theorem 2.12. *Assume that $(E \rightarrow M, \partial_D M)$ satisfies the 2-Poincaré inequality and use the notation of Lemma 2.6. In particular, $Pu = P_a u + Qu$ is assumed to be uniformly strongly elliptic.*

- (i) *If $c \in L^\infty(M; \text{End } E)$, $\Re c \geq R_\eta$ with R_η as in Lemma 2.6 for some $\eta \in (0, c_a)$, then $P + c: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism.*
- (ii) *If $c \in L^\infty(M; \text{End } E)$, $\Re c \geq 0$, and $Q + Q^*: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is of small enough norm (which is e.g. the case if $Q = 0$), then $P + c: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism.*

Note that (i) is very similar to Corollary 2.8, but there is no ϵ in the statement. Getting rid of the additional $\epsilon > 0$ makes it possible to use this result to study the Laplace-Beltrami operator Δ_g as in Example 2.2, in which case we can take $c = R_\eta = 0$. In contrast to Corollary 2.8, the results on Δ_g require the Poincaré inequality, which motivates our further study. Moreover, the assumption that $Q + Q^*$ has small enough norm will be quantified in the proof, in particular the bound will only depend on the coercivity constant c_a and on the constant c_2 in the 2-Poincaré inequality.

Proof. We proceed as in the proof of Corollary 2.8, using, in particular, the Lax–Milgram lemma. We have from Lemma 2.6 that $\Re \langle Pu, u \rangle \geq \eta |u|_{H^1}^2 - R_\eta \|u\|_{L^2}^2$ for any $u \in H_D^1(M; E)$. If $\Re c \geq R_\eta$, we obtain that $P_1 := P + c$ satisfies $\Re \langle P_1 u, u \rangle \geq \eta |u|_{H^1}^2 \geq \eta' \|u\|_{H^1}^2$, for some $\eta' > 0$, by Lemma 2.11. Therefore P_1 satisfies the assumptions of the Lax–Milgram lemma and hence it is invertible, as claimed. This proves (i).

To prove (ii), we proceed similarly. We have $Q: H_D^1(M; E) \rightarrow L^2(M; E)$, and hence $Q^*: L^2(M; E) \rightarrow H_D^1(M; E)^*$, so $Q + Q^*: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is defined. Thus, let $\|Q + Q^*\| < 2\epsilon$, where the norm is that of a bounded operator $H_D^1(M; E) \rightarrow H_D^1(M; E)^*$. Since a is bounded, we also have that the form B_a is

bounded. Next, the assumption on a and the Poincaré inequality for $p = 2$ together with Equation (17) give

$$\begin{aligned} \Re \langle (P + c)u, u \rangle &\geq c_a |u|_{H^1}^2 + \Re \langle Qu, u \rangle + \Re \langle cu, u \rangle \geq c_a |u|_{H^1}^2 + \frac{1}{2} \langle (Q + Q^*)u, u \rangle \\ &\geq c_a |u|_{H^1}^2 - \epsilon \|u\|_{H^1}^2 \geq \left(\frac{c_a}{1 + c_2^2} - \epsilon \right) \|u\|_{H^1}^2, \end{aligned}$$

for all $u \in H_D^1(M; E)$, where the last inequality is from Lemma 2.11. For ϵ small enough, the assumptions of the Lax–Milgram lemma are thus satisfied, and hence $P + c$ is an isomorphism. \square

If $\partial_D M = \partial M$, we also obtain the solvability of the Poisson problem with homogeneous Dirichlet boundary conditions. Recall that Δ_g denotes the Laplace–Beltrami operator associated to the metric g on M , see Example 2.2. We notice that the current definition of $H^{-1}(M; E) := H_0^1(M; E)^*$ as the (conjugate) dual of $H_0^1(M; E)$ with pivot $L^2(M, g)$ is compatible with other possible definitions of this space, such as the one using partitions of unity, as in Proposition 5.7.

Corollary 2.13. *We have that $(E \rightarrow M, \partial_D M)$ satisfies the 2-Poincaré inequality if, and only if, $P_a: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism for $a = g \otimes \langle \cdot, \cdot \rangle_E$. In particular, if $\partial_N M$ is empty, then the 2-Poincaré inequality holds on M if, and only if, $\Delta_g: H_0^1(M) \rightarrow H_0^1(M)^* =: H^{-1}(M)$ is an isomorphism.*

Proof. For simplicity, we shortly write P_g for $P_{g \otimes \langle \cdot, \cdot \rangle_E}$ and suppress the E -part in the indices of the associated norms. If the 2-Poincaré inequality holds, Theorem 2.12(ii) implies that $P_g: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism.

The converse is a consequence of some standard estimates. Indeed, let us assume that $P_g: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism. If the 2-Poincaré inequality is not satisfied, then there exists a sequence of sections $u_n \in H_D^1(M; E)$ such that $\|u_n\|_{H^1} = 1$, but $\|\nabla u_n\|_{L^2}^2 = \langle P_g u_n, u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$, by Lemma 2.11. Since P_g is an isomorphism, there is a $\gamma > 0$ such that $\|P_g u\|_{(H_D^1)^*} \geq 2\gamma \|u\|_{H^1}$. Therefore, using the definition of the norm on $H_D^1(M; E)^*$ we can find a sequence $v_n \in H_D^1(M; E)$ such that $\|v_n\|_{H^1(M; E)} = 1$ and $\langle P_g u_n, v_n \rangle \geq \gamma$. Using the Cauchy–Schwarz inequality for the L^2 -scalar product on E -valued 1-forms, we then obtain the following

$$\begin{aligned} \|\nabla u_n\|_{L^2}^2 &= \|\nabla u_n\|_{L^2}^2 \|v_n\|_{H^1}^2 \geq \|\nabla u_n\|_{L^2}^2 \|\nabla v_n\|_{L^2}^2 \\ &\geq |(\nabla u_n, \nabla v_n)|^2 = |\langle P_g u_n, v_n \rangle|^2 \geq \gamma^2 > 0, \end{aligned}$$

which contradicts $\|\nabla u_n\|_{L^2} \rightarrow 0$. \square

2.6. The Poincaré inequality: inclusions and submersions. The results of the previous subsection underscore the importance of the Poincaré inequality. In this subsection, we establish several easy reduction results leading to the Poincaré inequality for new manifolds out of the Poincaré inequality for some given manifold.

In the following we assume that Ω is a manifold with boundary $\partial\Omega \subset \Omega$, $\dim M = \dim \Omega$, and embedded as a closed subset of M . The following result shows that we may often assume $\partial_D \Omega$ to have low regularity.

Proposition 2.14. *Let us assume that Ω has a similar decomposition $\partial\Omega = \partial_D \Omega \sqcup \partial_N \Omega$ of the boundary as a disjoint union of two open subsets. Assume that $(E \rightarrow$*

$M, \partial_D M)$ satisfies the p -Poincaré inequality and that $\partial_N \Omega = \partial_N M$. Then $(E|_\Omega \rightarrow \Omega, \partial_N \Omega)$ satisfies the p -Poincaré inequality with the same constant.

Proof. We shall write simply E for $E|_\Omega$. Our assumption on the Neumann boundaries gives $\Gamma_c(\Omega \setminus \partial_D \Omega; E) \subset \Gamma_c(M \setminus \partial_D M; E)$. By taking completions, we obtain that $W_D^{1,p}(\Omega; E) \subset W_D^{1,p}(M; E)$ naturally. Thus, the inequality for $E \rightarrow \Omega$ then follows from the inequality for $E \rightarrow M$. \square

We shall assume for the rest of this subsection that M satisfies the following conditions for some fixed $p \in [1, \infty]$:

- (i) There exists a submersion $\pi: M \rightarrow L$, where L is a manifold *without* boundary.
- (ii) Let $M_x := \pi^{-1}(x)$ and let dvol_x denote the volume form on M_x . We assume that there is $C > 0$ such that, for each $x \in L$ and each $u \in \Gamma_c(M_x \setminus \partial_D M; E)$,

$$|u|_{W^{1,p}(M_x; E)} \geq C \|u\|_{L^p(M_x; E)}.$$

(That is, the pairs $(M_x, M_x \cap \partial_D M)$ satisfy the p -Poincaré inequality with a uniform bound in $x \in L$.)

- (iii) We assume given a measure $\mu \geq 0$ on L and a constant $c > 0$ satisfying

$$c^{-1} \int_M f \, \text{dvol}_g \leq \int_L \left(\int_{M_x} f \, \text{dvol}_x \right) d\mu \leq c \int_M f \, \text{dvol}_g, \quad 0 \leq f \in \Gamma_c(M \setminus \partial_D M).$$

Let $W := \cup_x TM_x = \ker \pi_* \subset TM$ be the longitudinal tangent bundle to the fibers of π .

Proposition 2.15. *Let $(E \rightarrow M, \partial_D M)$ satisfy (i)–(iii) above for $p = 2$ and let a be a bounded family of sesquilinear forms. Assume that there exists $\tilde{c} > 0$ satisfying $a(\xi, \xi) \geq \tilde{c} \|\xi|_W\|^2$ for all $\xi \in T^*M \otimes_{\mathbb{R}} E$. Then there exists $c_1 > 0$ such that $\langle P_a u, u \rangle \geq c_1 \|u\|_{L^2(M; E)}^2$.*

Proof. We have

$$\begin{aligned} \langle P_a u, u \rangle &= \int_M a(\nabla u, \nabla u) \, \text{dvol}_g \geq \tilde{c} c^{-1} \int_L \left(\int_{M_x} |\nabla u|_W^2 \, \text{dvol}_x \right) d\mu \\ &\geq C^2 \tilde{c} c^{-1} \int_L \left(\int_{M_x} |u|^2 \, \text{dvol}_x \right) d\mu \geq C^2 \tilde{c} c^{-2} \|u\|_{L^2}^2 \end{aligned}$$

\square

Results similar to the following one are often used in practice.

Proposition 2.16. *If $(E \rightarrow M, \partial_D M)$ satisfies conditions (i)–(iii), then it satisfies the p -Poincaré inequality. If $p = \infty$, then the assumption (iii) is not needed.*

Proof. Let $p < \infty$.

$$\begin{aligned} \int_M |\nabla u|^p \, \text{dvol}_g &\geq \int_M |\nabla u|_W^p \, \text{dvol}_g \geq c^{-1} \int_L \left(\int_{M_x} |\nabla u|_W^p \, \text{dvol}_x \right) d\mu \\ &\geq C^p c^{-1} \int_L \left(\int_{M_x} |u|^p \, \text{dvol}_x \right) d\mu \geq C^p c^{-2} \|u\|_{L^p}^p \end{aligned}$$

For $p = \infty$ the argument is analogous. \square

3. MANIFOLDS WITH BOUNDARY AND BOUNDED GEOMETRY AND STATEMENT OF THE POINCARÉ INEQUALITY

In this section, we provide a large class of manifolds for which we prove the Poincaré inequality directly. We introduce manifolds with boundary, respectively vector bundles of bounded geometry, we state our Poincaré inequality for manifolds with finite width, and then we prove some preliminary results for such manifolds in the special case where the metric is a product near the boundary. We continue to assume that M is a Riemannian manifold with *smooth* boundary $\partial M = \partial_D M \sqcup \partial_N M$.

3.1. Manifolds with boundary and bounded geometry. For any $x \in M$ we define

$$\mathcal{D}_x := \{v \in T_x M \mid \text{there exists a geodesic } \gamma_v : [0, 1] \rightarrow M \text{ with } \gamma'_v(0) = v\}.$$

This set is open and star-shaped in $T_x M$ and

$$\mathcal{D} := \bigcup_{x \in M} \mathcal{D}_x$$

is an open subset of TM . The map $\exp^M : \mathcal{D} \rightarrow M$, $\mathcal{D}_x \ni v \mapsto \exp^M(v) = \exp_x^M(v) := \gamma_v(1)$ is called the *exponential map*.

If $x, y \in M$, then $\text{dist}(x, y)$ denotes the distance between x and y , computed as the infimum of the set of lengths of the paths in M connecting x to y . If $N \subset M$ is a subset, then

$$U_r(N) := \{x \in M \mid \exists y \in N, \text{dist}(x, y) < r\}$$

will denote the r -neighborhood of N , that is, the set of points of M at distance $< r$ to N . Thus, if E is a Euclidean space, then $B_r^E(0) := U_r(\{0\}) \subset E$ is simply the ball of radius r centered at 0.

Let N be a hypersurface in M , i.e. a submanifold with $\dim N + 1 = \dim M$. We assume that N carries a globally defined normal vector field ν of unit length, briefly called a *unit normal field*, which allows us to identify the normal bundle of N in M with $N \times \mathbb{R}$. The Levi-Civita connection for the induced metric on N is called ∇^N . The symbol Π will denote the second fundamental form of N . Recall then that Π is the smooth family of symmetric bilinear maps $\Pi_p : T_p N \times T_p N \rightarrow \mathbb{R}$, $p \in N$, defined by

$$\Pi(X, Y)\nu := \nabla_X Y - \nabla_X^N Y$$

for any two tangent vector fields X and Y of N . In particular, we see that Π defines a smooth tensor. See [27, Chap. 6] for details.

Let \mathcal{D}^\perp be the intersection of \mathcal{D} with the normal bundle of N in M and

$$(19) \quad \exp^\perp := \exp|_{\mathcal{D}^\perp} : \mathcal{D}^\perp \rightarrow M, \quad \exp^\perp(x, t) := \exp_x^M(t\nu_x)$$

be the normal exponential map. Using the identification above $\mathcal{D}^\perp \subset N \times \mathbb{R}$. As \mathcal{D} is open in TM , we know that \mathcal{D}^\perp is a neighborhood of $N \times \{0\}$ in $N \times \mathbb{R}$.

Let

$$r_{\text{inj}}(p) := \sup\{r \mid \exp_p : B_r^{T_p M}(0) \rightarrow M \text{ is a diffeomorphism onto its image}\}$$

$$r_{\text{inj}}(M) := \inf_{p \in M} r_{\text{inj}}(p).$$

Definition 3.1. Recall that a (real or complex) vector bundle E with given connection has *totally bounded curvature* if its curvature R^E satisfies

$$\|\nabla^k R^E\|_{L^\infty} < \infty \quad \text{for all } k \in \mathbb{N}_0.$$

Note that to formulate this condition the Riemannian metric and the Levi-Civita on T^*M is implicitly used. For $E = TM$ with the Levi-Civita connection, we write $R^M := R^{TM}$. We say that M has *totally bounded curvature*, if TM has totally bounded curvature.

Recall the following classical definition in the case $\partial M = \emptyset$:

Definition 3.2. A Riemannian manifold without boundary (M, g) is said to be of *bounded geometry* if $r_{\text{inj}}(M) > 0$ and if M has *totally bounded curvature*.

This definition cannot be carried over in a straightforward way to manifolds with boundary, as manifolds with non-empty boundary always have $r_{\text{inj}}(M) = 0$.

Definition 3.3. Let (M^m, g) be a Riemannian manifold of bounded geometry with a hypersurface $N^{m-1} \subset M$ and a unit normal field. We say that N is a *bounded geometry hypersurface* if the following is fulfilled

- (i) N is a closed subset of M .
- (ii) $(N, g|_N)$ is a manifold of bounded geometry.
- (iii) The second fundamental form Π of N in M and all its covariant derivatives along N are bounded. In other words $\|(\nabla^N)^k \Pi\| \leq C_k$ for all $k \in \mathbb{N}_0$.
- (iv) There is a number $\delta > 0$ such that $\exp^\perp: N \times (-\delta, \delta) \rightarrow M$ is injective.

We will prove in Section 6 that Axiom (ii) is redundant, i.e. it already follows from the other axioms. Nevertheless, we decided to keep Axiom (ii) here to make the comparison of our definition with the definitions in [58] and [34] easier.

Definition 3.4. A Riemannian manifold M with (smooth) boundary has *bounded geometry* if there is a Riemannian manifold \widehat{M} with bounded geometry satisfying

- (i) $\dim \widehat{M} = \dim M$
- (ii) M is contained in \widehat{M}
- (iii) ∂M is a bounded geometry hypersurface in \widehat{M} .

As unit normal vector field for ∂M we choose the outer unit normal field. We will show in Section 6 that this definition coincides with [58, Definition 2.2]. In this section we also discuss further conditions which are equivalent to the axioms in Definition 3.4.

Note that if M is a manifold with boundary and bounded geometry, then M is a complete metric space.

Example 3.5. An important example for a manifold with boundary and bounded geometry is provided by Lie manifolds with boundary, [7, 6]. In particular, Lie manifolds are manifolds of bounded geometry.

For the Poincaré inequality, we shall also need to assume that $M \subset U_R(\partial_D M)$, for some $R > 0$, and hence, in particular, that $\partial_D M \neq \emptyset$.

Definition 3.6. If M is a manifold with boundary and bounded geometry, if $\partial_D M$ is an open component of the boundary ∂M , and $M \subset U_R(\partial_D M)$, for some $R > 0$, we shall say that $(M, \partial_D M)$ has *finite width*. If $\partial_D M = \partial M$, we shall also say that M has *finite width*.

Note that the meaning of the assumption that $M \subset U_R(\partial_D M)$ in this definition is that every point of M is at distance *at most* R to $\partial_D M$.

Example 3.7. A very simple example of a manifold with finite width is obtained by considering a smooth, connected, compact manifold Ω with smooth boundary and a closed Riemannian manifold K . Then K has bounded geometry. Let $\partial_D \Omega$ be a non-empty union of connected components of $\partial \Omega$. Let $M := \Omega \times K$ and $\partial_D M := \partial_D \Omega \times K$. Then $(M, \partial_D M)$ is a manifold with finite width.

On the other hand, let $\Omega \subset \mathbb{R}^N$ be an open subset with smooth boundary. We assume that the boundary of Ω coincides with the boundary of a cone outside some compact set. Then $(\Omega, \partial \Omega)$ is a manifold with boundary and bounded geometry, but it does not have finite width.

Definition 3.8 ([29, Section 1.A.1]). Let $(E, \nabla^E, \langle \cdot, \cdot \rangle_E)$ be a hermitian vector bundle over a Riemannian manifold (M, g) with boundary. In particular, ∇^E is supposed to preserve the Hermitian form. We say that $(E, \nabla^E, \langle \cdot, \cdot \rangle_E)$ is of bounded geometry, if (M, g) is of bounded geometry and E has totally bounded curvature.

Remark 3.9. See [34, Section 5.1] for equivalent definitions of vector bundles of bounded geometry.

3.2. Statement of our Poincaré inequality. Recall that we have decomposed the boundary into two closed, disjoint submanifolds $\partial_D M$ and $\partial_N M$. For our Poincaré inequality, which we formulate next, we will also have $\partial_D M \neq \emptyset$. Note that Assumption (14) is satisfied in view of the results of [34] (see Theorem 5.9), that is $H_D^1(M; E) = \{u \in H^1(M; E) : u|_{\partial_D M} = 0\}$.

Theorem 3.10 (Poincaré inequality). *Let M be an m -dimensional smooth Riemannian manifold with smooth boundary $\partial M = \partial_D M \sqcup \partial_N M$ and with bounded geometry. Let $E \rightarrow M$ be a vector bundle with bounded geometry. Assume that $(M, \partial_D M)$ has finite width, i.e. $M \subset U_R(\partial_D M)$, for some $R \in (0, \infty)$. For every $p \in [1, \infty]$ there exists $0 < c = c_{M,p} < \infty$ such that*

$$\|f\|_{L^p} \leq c \|\nabla f\|_{L^p}$$

for all $f \in H_D^1(M; E)$.

In particular, if $\|\nabla f\|_{L^p}$ is finite, then $\|f\|_{L^p}$ is also finite, and the inequality holds. The proof of this result will be split into several steps. In a first step, carried out in Subsection 4.1, we will prove the Poincaré inequality under the additional assumption that we have a Riemannian product structure close to the boundary. For this purpose, some geometric preliminaries will be provided in Subsection 3.3 under the product assumption. Then the general case will follow in Subsection 4.2. See also [20, 37, 57] for Poincaré type inequalities on manifolds without boundary and bounds on the Ricci tensor.

We assume from now on and throughout the paper that M is a manifold with boundary and bounded geometry. We always assume $M \subset U_R(\partial_D M)$ for the fixed R as above. Unless mentioned otherwise, the boundary ∂M may be empty.

As we will see below, introducing the vector bundle E , while important, affects very little the proof.

3.3. Geometric preliminaries for manifolds with product metric. In the following subsection, we will assume additionally that there exists $r_\partial > 0$ such that

the metric g is a product metric near the boundary, that is, it has the form

$$(20) \quad g = g_{\partial} + dt^2, \quad \text{on } \partial M \times [0, r_{\partial}),$$

where g_{∂} is the metric induced by g on ∂M .

We identify in what follows $\partial M \times \mathbb{R}$ with the normal bundle to ∂M in M as before for hypersurfaces, i.e. we identify $(x, t) \in \partial M \times \mathbb{R}$ with $t\nu_x$ in the normal bundle of ∂M in M . Also, we shall identify $(x, t) \in \partial M \times [0, r_{\partial})$ with $\exp^{\perp}(x, t) = \exp^{\perp}(t\nu_x) \in M$.

Our proof of the Poincaré inequality under the product metric assumption on $\partial M \times [0, r_{\partial})$ is based on several intermediate results. By decreasing r_{∂} we may assume that δ in Definition 3.3.(iv) and r_{∂} in Equation (20) are the same.

First, note that, by the product structure of g near the boundary, the submanifolds $\partial M \times \{t\}$ with $t \in [0, r_{\partial})$ are totally geodesic submanifolds of M and that a geodesic in $\partial M \times [0, r_{\partial})$ always has the form $c(t) = \exp^{\perp}(c_{\partial}(t), at)$ for some $a \in \mathbb{R}$ and some geodesic c_{∂} in $(\partial M, g_{\partial})$. This implies that a geodesic $c: [a, b] \rightarrow M$ with $c(a) \notin \partial M$ and $c(b) \in \partial M$ cannot be extended to $[a, b + \epsilon]$ for any $\epsilon > 0$.

Recall that for a Riemannian manifold, the distance $\text{dist}(x, y)$ for $x, y \in M$ is defined as the infimum of the lengths of all rectifiable curves joining these points. For a subset $A \subset M$ we define

$$\text{dist}(y, S) := \inf_{x \in S} \{\text{dist}(x, y)\}.$$

Let $y \in M$. A *shortest curve joining y to $S \subset M$* is by definition a rectifiable curve $\gamma: [a, b] \rightarrow M$ from y to S (that is, $\gamma(a) = y$, $\gamma(b) \in S$) such that no other curve from y to S is shorter than γ . If a shortest curve is parametrized proportional to arc length and its interior does not intersect the boundary, then it is a geodesic. Such geodesics will be called *length minimizing geodesics*.

Proposition 3.11. *Let M have product structure near the boundary. For every $y \in M \setminus \partial_N M$, there is a length minimizing smooth geodesic γ from y to $\partial_D M$.*

In general, there may be more than one (geometrically distinct) shortest geodesics $\gamma: [a, b] \rightarrow M$ joining y to $\partial_D M$. If $y \notin \partial_D M$, every such curve is such that $\gamma(t)$ is in the interior of M for all $t \in [a, b)$ and such that $\gamma'(b)$ is perpendicular to $\partial_D M$.

Proof. The proof is analogous to the classical proof of the Hopf-Rinow Theorem (which states that in a geodesically complete manifold, any two points are joined by a length minimizing geodesic, see Chapter 7 in [27]). We follow Theorem 2.8 (Hopf-Rinow) in Chapter 7 of the aforementioned book. For a given point $y \in M \setminus \partial_N M$ we define $r := \text{dist}(y, \partial_D M)$. We only have to consider the case $y \notin \partial_D M$, and in this case $r > 0$. It follows from the Gauss lemma that the length of any curve joining y to a point of the “sphere” $S_{\delta}(y) := \{\exp_y(\delta v) \mid |v| = 1\}$ is at least δ , with the infimum being attained by the image of the straight line under \exp_y , provided that $\delta < r_{\text{inj}}(M)$, see Chapter 3 in [27] for details. The function $x \mapsto \text{dist}(x, \partial_D M)$ is continuous and thus we can choose a point $x_0 \in S_{\delta}(y)$ with

$$\text{dist}(x_0, \partial_D M) = \min\{\text{dist}(x, \partial_D M) \mid x \in S_{\delta}(y)\}.$$

Every curve from y to $\partial_D M$ will hit $S_{\delta}(y)$ somewhere, thus we obtain

$$(21) \quad \text{dist}(y, x_0) + \text{dist}(x_0, \partial_D M) = \text{dist}(y, \partial_D M).$$

Let $|v| = 1$ with $\exp_y(\delta v) = x$. We now claim that

$$(22) \quad \exp_y(tv) \text{ is defined and } \text{dist}(\exp_y(tv), \partial_D M) = r - t$$

holds for all $t \in (0, r)$. The proof is again analogous to the proof of the theorem by Hopf-Rinow in [27]. Let $A := \{t \in [0, r] \mid (22) \text{ holds for } t\}$. Obviously A is closed in $[0, r]$, and from the triangle inequality we see that $t \in A$ implies $\tilde{t} \in A$ for all $\tilde{t} \in [0, t]$. So $A = [0, b]$ for some $b \in [0, r]$. Further (21) implies $b \geq \delta$. Due to the product structure near $\partial_N M$ the geodesic $[0, b] \ni t \mapsto \exp_y(tv)$ does not hit $\partial_N M \times [0, d(y))$ for some $d(y)$ small enough, as otherwise this would violate (22). We will show that for any $s_0 \in A$, $s_0 < r$ there is a $\delta' > 0$ with $s_0 + \delta' \in A$. In this goal we repeat the above argument for $y' := \exp_y(s_0 v)$ instead of y . We obtain $\delta' > 0$ and $x'_0 \in S_{\delta'}(y')$ such that $\text{dist}(y', x'_0) + \text{dist}(x'_0, \partial_D M) = \text{dist}(y', \partial_D M)$, and we write $x'_0 = \exp_{y'}(\delta' v')$. This implies

$$(23) \quad \text{dist}(y, y') + \text{dist}(y', x'_0) + \text{dist}(x'_0, \partial_D M) = \text{dist}(y, \partial_D M),$$

and then we get $\text{dist}(y, y') + \text{dist}(y', x'_0) = \text{dist}(y, x'_0)$. We have shown that the curve

$$\gamma(t) := \begin{cases} \exp_y(tv) & 0 \leq t \leq s_0 \\ \exp_{y'}((t - s_0)v') & s_0 \leq t \leq s_0 + \delta' \end{cases}$$

is a shortest curve from y to x'_0 and thus the geodesic is not broken in y' in other words

$$\exp_y(tv) = \exp_{y'}((t - s_0)v') \quad s_0 \leq t \leq s_0 + \delta'.$$

Using (23) once again, we see that $s_0 + \delta' \in A$.

We have seen that $b = \max A = r$. We obtain $\exp_y(rv) \in \partial_D M$ which gives the claim. Moreover, the first variation formula implies $\gamma'(r) \perp \partial_D M$. \square

Proposition 3.12. *There is a continuous function $L: \partial_D M \rightarrow (0, \infty]$ such that the restriction of \exp^\perp to*

$$\{(x, t) \mid 0 \leq t \leq L(x), x \in \partial_D M\}$$

is surjective, and such that the restriction of \exp^\perp to

$$\{(x, t) \mid 0 < t < L(x), x \in \partial_D M\}$$

is an embedding. Furthermore

$$\text{dist}(\exp^\perp(x, t), \partial_D M) = t$$

if $0 \leq t \leq L(x)$. The set

$$M_S := \exp^\perp(\{(x, t) \mid t = L(x), x \in \partial_D M\})$$

is of measure zero.

Proof. For $x \in \partial_D M$, let us consider the geodesic $\gamma_x: I_x \subset [0, \infty) \rightarrow M$ with $\gamma_x(0) = x$ and $\gamma'_x(0) = \nu_x$, for $x \in \partial_D M$, defined on its maximal domain I_x . We choose $L(x) \in I_x$ as the maximal number such that $\gamma_x|_{[0, L(x)]}$ realizes the minimal distance from x to $\gamma_x(t)$ if $0 \leq t \leq L(x)$. Let $y \in M$ and $d = \text{dist}(y, \partial_D M)$. From the Proposition 3.11 above we see that a shortest curve from y to $\partial_D M$ exists; in other words there is $x \in \partial_D M$ with $y = \exp^\perp(x, d)$ where $\exp^\perp(x, t) := \exp_x(t\nu_x)$. Therefore, the restriction of \exp^\perp to $\{(x, t) \mid 0 \leq t \leq L(x), x \in \partial_D M\}$ is surjective. The continuity of L is analogous to [27, Chap. 13, Prop. 2.9]. As the geodesics $t \mapsto \gamma_x(t)$ are minimizing for $0 \leq t \leq L(x)$, it follows similar to [27, Chap. 13, Prop. 2.2] that there is a unique shortest curve from $\gamma_x(t)$ to $\partial_D M$ if $0 \leq t < L(x)$, and that the restriction of \exp^\perp to $\{(x, t) \mid 0 < t < L(x), x \in \partial_D M\}$ is an injective

immersion. From the inverse function theorem we see that this injective immersion is a homeomorphism on its image, thus it is an embedding.

The subset M_S of M is closed and has measure zero as it is the image of the measure zero set $\{(x, L(x)) \mid x \in \partial_D M\}$ under the smooth map \exp^\perp . \square

Remark 3.13. One usually defines the *cut locus* $\mathcal{C}(S)$ of a subset $S \subset M$ as the set of all points x in the interior of M for which there is a geodesic $\gamma: [-a, \epsilon) \rightarrow M$ with $\gamma(-a) \in S$, $\gamma(0) = x$, γ being minimal for all $t \in (-a, 0)$, but no longer minimal for $t > 0$. The name “cut locus” comes from the fact that this the set where either several shortest curves emanating from S will either intersect classically or in an infinitesimal sense. The relevance of this concept is that the set M_S introduced in Proposition 3.12 satisfies $M_S = \partial_N M \cup \mathcal{C}(\partial_D M)$.

If $H: T_p M \rightarrow T_q M$ is an endomorphism, then we express it in orthonormal bases as a matrix A . The definition

$$(24) \quad |\det H| := |\det A|$$

is well-defined and does not depend on the choice of orthonormal bases. Note that $\det H$ would be well-defined only after fixing orientations. Similarly, if v_1, \dots, v_m is a basis of a Euclidean vector space, then we choose an orthonormal basis e_1, \dots, e_m and write $v_j = \sum_i a_{ij} e_i$ and define

$$(25) \quad |\det(v_1, \dots, v_m)| := |\det(a_{ij})|$$

which is independent of the choice of orthonormal basis.

For $x \in \partial_D M$, $0 \leq t \leq L(x)$, let $v(x, t)$, be the volume distortion of the normal exponential map, that is,

$$(26) \quad v(x, t) := |\det d_{(x,t)} \exp^\perp|,$$

with the absolute value of the determinant defined using local orthonormal bases, as in (24).

Proposition 3.14. *We continue to assume that M^m is a manifold with boundary and bounded geometry and that the metric is a product near the boundary. We also assume that there exists $R > 0$ with $\text{dist}(x, \partial_D M) < R$ for all $x \in M$. Then there is a constant $C > 0$ such that, for all $x \in \partial M$ and all $0 \leq s \leq t \leq L(x)$, we have*

$$\frac{v(x, t)}{v(x, s)} \leq C.$$

The constant C can be chosen to be $e^{(m-1)R\sqrt{|c|}}$ where $(m-1)c$ is a lower bound for the Ricci curvature of M .

This proposition is essentially a special case of the Heintze–Karcher inequality [41]. We refer to the section on Heintze–Karcher inequalities in [10] for a proof of the full statement, some historical notes, and some similar inequalities. For the benefit of the reader we include here a proof of the proposition.

Proof. Let $x \in \partial_D M$ and let $\{e_1, \dots, e_{m-1}, \nu_x\}$ be an orthonormal frame for $T_x M$. Let $\gamma_i(s) := \exp_x^M(s e_i)$ and $c_i(t, s) := \exp_{\gamma_i(s)}^M(t \nu_{\gamma_i(s)})$. We obtain

$$d_{(x,t)} \exp^\perp(e_i) = \frac{d}{ds} \Big|_{s=0} \underbrace{\exp_{\gamma_i(s)}^M(t \nu_{\gamma_i(s)})}_{=: c_i(t,s)}.$$

Thus, $c_i(t, s)$ is a geodesic variation of $t \mapsto c_0(t) = c_i(t, 0)$ with variational parameter s and variation vector field $d_{(x,t)} \exp^\perp(e_i) = J_i(t)$ where $J_i(t) = \frac{\partial c_i}{\partial s}(t, 0)$ is the unique Jacobi field along c_0 with initial values $J_i(0) = \frac{\partial c_i}{\partial s}(0, 0) = \gamma'_i(0) = e_i$ and $(\nabla_{\frac{\partial c_i}{\partial t}} J_i)(0, 0) = (\nabla_{\nu_{\gamma_i(s)}} \gamma'_i(s))(0, 0) = (\nabla_{\gamma'_i(s)} \nu_{\gamma_i(s)})(0, 0) = 0$. The Jacobi fields $J_i(t)$ are normal to $c'_0(t)$, cf. [27, Chap. 5]. Moreover,

$$d_{(x,t)} \exp^\perp(\nu_x) = \frac{d}{ds}|_{s=0} \exp_x^M((t+s)\nu_x).$$

Thus, $|d_{(x,t)} \exp^\perp(\nu_x)| = 1$. Putting all this together, we obtain

$$|\det d_{(x,t)} \exp^\perp| = |\det(J_1(t), \dots, J_{m-1}(t))|,$$

where the right hand side is defined using (25) and where the vectors $J_i(t)$ are viewed as vectors in

$$T_{c_0(t)}^\perp M := \{X \in T_{c_0(t)} M \mid X \perp c'_0(t)\}.$$

Proposition 3.12 tells us that the function $\phi(x) := \text{dist}(x, \partial_D M)$ is smooth on $M \setminus M_S$, and the above considerations yield $|d\phi| = 1$ on this subset. Such functions are called *generalized distance functions* and the submanifolds $\phi^{-1}(t)$ are submanifolds for $s \in (0, R)$. The gradient $\text{grad } \phi$ of ϕ is a unit normal field of each such hypersurface.

Let Π^t be the second fundamental form of the submanifold $\phi^{-1}(t)$ with respect to $\text{grad } \phi$. The corresponding Weingarten map S_t is then defined by $\Pi^t(X, Y) = g(S_t(X), Y)$.

Then $\nabla_{c'_0(t)} J(t) = -S_t(J(t))$. With the notation

$$A_t := (J_1(t), \dots, J_{m-1}(t)) \in \text{Hom}(\mathbb{R}^{m-1}, T_{c_0(t)}^\perp M)$$

this yields

$$\nabla_{c'_0(t)} A_t = -S_t \circ A_t.$$

We get

$$\begin{aligned} \frac{\partial}{\partial t} |\det A_t| &= |\det A_t| \text{tr}((\nabla_{c'_0(t)} A_t) \circ A_t^{-1}) \\ &= -|\det A_t| \text{tr } S_t \end{aligned}$$

Thus, we obtain for the mean curvature $H_t = (\text{tr } S_t)/(m-1)$ of the level sets $\phi^{-1}(t)$ that

$$(27) \quad H_t = -\frac{1}{m-1} \frac{d}{dt} \log |\det A_t|.$$

From the Riccati equation [41], see also [31] we have $\partial_t H_t \geq \frac{\text{ric}(\text{grad } \phi, \text{grad } \phi)}{m-1} + H_t^2$. Thus bounded geometry implies $\partial_t H_t \geq c + H_t^2$ for some global constant $c \in \mathbb{R}_{<0}$. Furthermore H_t vanishes on $\partial_D M$, i.e. for $t = 0$. From Riccati's comparison theorem, cp. [41, Sec. 1.5 and 1.6], we obtain $H_t \geq h(t) := -\sqrt{|c|} \tanh(\sqrt{|c|}t) \geq -\sqrt{|c|}$.

Together with (27) we obtain

$$\frac{d}{dt} \log |\det A(t)| = -(m-1)H_t \leq (m-1)\sqrt{|c|},$$

and this implies

$$\frac{v(x, t)}{v(x, s)} = \frac{|\det A_t|}{|\det A_s|} \leq e^{(t-s)(m-1)\sqrt{|c|}} < C$$

for a universal constant $C \in \mathbb{R}$. In the very last inequality we used that $t - s < R < \infty$. \square

4. PROOF OF THE POINCARÉ INEQUALITY

In this section, we provide a proof of the Poincaré inequality stated in Theorem 3.10. We deal first with the case when the metric is a product near the boundary, as in the last subsection. Then we show that the general case can be reduced to the product case near the boundary. Recall that, for the rest of this paper, M is a manifold with boundary and bounded geometry. For the Poincaré inequality, we will also need to assume that M is of finite width.

4.1. The case of a manifold with product boundary. We keep all the notations introduced in the last subsection, Subsection 3.3. In particular, we assume as in the previous subsection that g is a product metric near the boundary.

Proof of Theorem 3.10 for g a product near the boundary. We set

$$\gamma_x(s) := \exp^\perp(x, s) := \exp_x^M(s\nu_x),$$

to simplify the notation. Recall that dvol_g denotes the volume element on M associated to the metric g . Let us assume first that $p < \infty$. Since

$$(28) \quad \int_M |u|^p \text{dvol}_g = \int_{\partial_D M} \int_0^{L(x)} |u(\gamma_x(s))|^p v(x, s) ds \text{dvol}_{g_\partial},$$

and $|\nabla u| \geq |\nabla_{\gamma'_x(s)} u|$, it suffices to find a $c > 0$ (independent on x) such that

$$(29) \quad \int_0^{L(x)} |\nabla_{\gamma'_x(s)} u|^p v(x, s) ds \geq c \int_0^{L(x)} |u(\gamma_x(t))|^p v(x, t) dt$$

for all $x \in \partial_D M$. Indeed; Equations (28) and (29) yield Theorem 3.10 by integration over $\partial_D M$ (recall that for $p < \infty$).

Let now $f(s) := u(\gamma_x(s))$, for some fixed $x \in M \setminus (M_S \cup \partial_D M)$. Then $f'(s) = \nabla_{\gamma'_x(s)} u$. Now we fix $t \in [0, L(x)]$. Let q be the exponent conjugate to p , that is, $p^{-1} + q^{-1} = 1$. Using $f(t) = \int_0^t f'(s) ds$, we obtain for $p < \infty$

$$\begin{aligned} |f(t)|^p v(x, t) &\leq \left(\int_0^t |f'(s)| ds \right)^p v(x, t) \leq t^{p/q} \int_0^t |f'(s)|^p v(x, t) ds \\ &\leq CL(x)^{p/q} \int_0^{L(x)} |f'(s)|^p v(x, s) ds \leq CR^{p/q} \int_0^{L(x)} |\nabla_{\gamma'_x(s)} u|^p v(x, s) ds. \end{aligned}$$

Hence, integrating once more with respect to t from 0 to $L(x) \leq R$, we obtain

$$CR^p \int_0^{L(x)} |\nabla_{\gamma'_x(s)} u|^p v(x, s) ds \geq \int_0^{L(x)} |f(t)|^p v(x, t) dt,$$

which gives (29) right away, and hence our result for $p < \infty$. The case $p = \infty$ is simpler. Indeed, it suffices to use instead

$$|f(t)| \leq \int_0^t |f'(s)| ds \leq t \|f'\|_{L^\infty} \leq R \|\nabla u\|_{L^\infty}$$

By taking the ‘sup’ on the left hand side, we obtain the result. \square

4.2. The general case. We now show how the general case of the Poincaré inequality for a metric with bounded geometry and finite width on M can be reduced to the case when the metric is a product metric in a small tubular neighborhood of the boundary, the case for which we have already proved the Poincaré inequality. We first introduce *Fermi coordinates* on M following [34]. See especially Definition 20 of that paper, whose notation we follow here. Recall that $r_{\text{inj}}(M)$ and $r_{\text{inj}}(\partial M)$ denote, respectively, the injectivity radii of M and ∂M . Also, let $r_\partial := \delta$ with δ as in Definition 3.3.

Let $p \in \partial M$ and consider the diffeomorphism $\exp_p^{\partial M}: B_r^{T_p \partial M}(0) \rightarrow B_r^{\partial M}(p)$, if r is smaller than the injectivity radius of ∂M . Sometimes, we shall identify $T_p \partial M$ with \mathbb{R}^{m-1} using an orthonormal basis, thus obtaining a diffeomorphism $\exp_p^{\partial M}: B_r^{m-1}(0) \rightarrow B_r^{\partial M}(p)$ where $B_r^{m-1}(0) \subset \mathbb{R}^{m-1}$ denotes the Euclidean ball around $0 \in \mathbb{R}^{m-1}$ and with radius r . Also, recall the definition of the normal exponential map $\exp^\perp: \partial M \times [0, r_\partial) \rightarrow M$, $\exp^\perp(x, t) := \exp_x^M(t\nu_x)$. These two maps combined and together with the exponential \exp_p^M define maps

$$(30) \quad \begin{cases} \kappa_p: B_r^{m-1}(0) \times [0, r) \rightarrow M, & \kappa_p(x, t) := \exp^\perp(\exp_p^{\partial M}(x), t), & \text{if } p \in \partial M \\ \kappa_p: B_r^m(0) \rightarrow M, & \kappa_p(v) := \exp_p^M(v), & \text{otherwise.} \end{cases}$$

In both cases, we let

$$(31) \quad U_p(r) := \begin{cases} \kappa_p(B_r^{m-1}(0) \times [0, r)) \subset M & \text{if } p \in \partial M \\ \kappa_p(B_r^m(0)) = \exp_p^M(B_r^m(0)) & \text{otherwise.} \end{cases}$$

In the next definition we need to consider only the case $p \in \partial M$, however, the other case will be useful in the next section when considering partitions of unity.

Definition 4.1. Let $p \in \partial M$ and $r_{FC} := \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_\partial \}$ and fix $0 < r \leq r_{FC}$. The map $\kappa_p: B_r^{m-1}(0) \times [0, r) \rightarrow U_p(r)$ is called a *Fermi coordinate chart* and the resulting coordinates $(x^i, r): U_p(r) \rightarrow \mathbb{R}^{m-1} \times [0, \infty)$ are called *Fermi coordinates (around p)*.

Figure 1 below describes the Fermi coordinate chart.

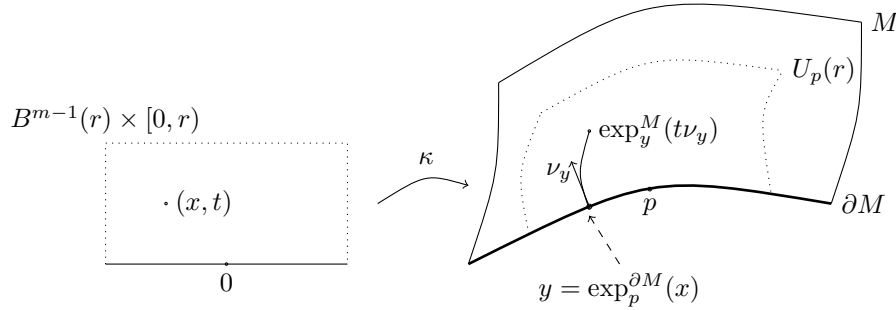


FIGURE 1. Fermi coordinates

Remark 4.2. Let (M, g) be a manifold with boundary and bounded geometry. Then the coefficients of g all their derivatives are uniformly bounded in Fermi coordinates charts, see, for instance, [34, Lemma 15 and Theorem 26].

The general case of Theorem 3.10 follows directly from the special case where g is product near the boundary and the following lemma. Recall that we identify $\partial M \times [0, r_\partial)$ with its image in M via the normal exponential map \exp^\perp .

Lemma 4.3. *Let (M^m, g) be a manifold with boundary and bounded geometry. Let g_∂ be the induced metric on the boundary ∂M . Then there is a metric g' on M of bounded geometry such that*

- (1) $g' = g_\partial + dt^2$ on $\partial M \times [0, r']$ for some $r' \in (0, r_\partial)$
- (2) g and g' are equivalent, that is, there is $C > 0$ such that $C^{-1}g \leq g' \leq Cg$.

In particular, the norms $|\cdot|_g$ and $|\cdot|_{g'}$ on E -valued one-forms, respectively on the volume forms for g and g' , are equivalent.

Proof. Let $0 < r' < r_{FC}/4 := \frac{1}{4} \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_\partial \}$, small enough, to be specified later. Here r_{FC} is as in the choice of our Fermi coordinates in Definition 4.1. Let $\eta: [0, 3r'] \rightarrow [0, 1]$ be a smooth function with $\eta|_{[0, r']} = 0$ and $\eta|_{[2r', 3r']} = 1$. We set $g'(x, t) := \eta(t)g(x, t) + (1 - \eta(t))(g_\partial(x) + dt^2)$ for $(x, t) \in \partial M \times [0, 3r']$ and $g' = g$ outside $\partial M \times [0, 3r']$. By construction g' is smooth and is a product metric on $\partial M \times [0, r']$. The rest of the proof is based on the use of Fermi coordinates around any $p \in \partial M$ to prove that g and g' are equivalent for r' small enough.

Then, in the Fermi coordinates $(x, t) := \kappa_p^{-1}$ around $p \in \partial M$, see Equation (30), we have $g_{ij}(x, t) = g_{ij}(x, 0) + tg_{ij,t}(x, 0) + O(t^2)$, $g_{it}(x, 0) = 0$ for $i \neq t$, $g_{tt}(x, t) = 1$, and $(g_\partial)_{ij}(x) = g_{ij}(x, 0)$ for $i, j \neq t$. Thus,

$$(32) \quad g'_{ij}(x, t) - g_{ij}(x, t) = \begin{cases} -(1 - \eta(t))(tg_{ij,t}(x, 0) + O(t^2)) & \text{if } (i, j) \neq (t, t) \\ 0 & \text{otherwise.} \end{cases}$$

Since $g_{ij,t}$ is uniformly bounded by Remark 4.2, we obtain $|g'_{ij}(x, t) - g_{ij}(x, t)| \leq tC$ for $(x, t) \in B_{r'}^{m-1} \times [0, r']$ where the constant C is independent of the chosen p . We note that in these coordinates the metric g is equivalent to the Euclidean metric on $B_{r'}^{m-1}(0) \times [0, r'] \subset \mathbb{R}^m$ in such a way that the equivalence constants do not depend on the chosen p . This can be seen from

$$\begin{aligned} |g_{ij}(x, t) - g_{ij}(0, 0)| &= |g_{ij}(x, t) - \delta_{ij}| \\ &\leq \sup |\nabla_{(x,t)} g_{ij}(x, t)| \|(x, t)\| + O(t^2) \leq Cr' + O(t^2), \end{aligned}$$

where C is the uniform bound for $\nabla_{(x,t)} g_{ij}(x, t)$, which is finite by the bounded geometry assumption. Moreover, $O(t^2) \leq ct^2$ with c depending on the uniform bound of the second derivatives on $g_{ij}(x, t)$ and r_{FC} . Let X be a vector in (x, t) . Then

$$|g(X, X) - |X|^2| = \left| \sum_{ij} (g_{ij} - \delta_{ij}) X^i X^j \right| \leq Cr' |X|^2.$$

Thus, for r' such that $Cr' < 1$, it follows that g and the Euclidean metric are equivalent on the chart around p such that the constants do not depend on p . Similarly, we then obtain

$$\begin{aligned} |g'(X, X) - g(X, X)| &= \left| \sum_{ij} (g'_{ij}(x, t) - g_{ij}(x, t)) X^i X^j \right| \\ &\leq r'C |X|^2 \leq r'C(1 - Cr')^{-1} g(X, X). \end{aligned}$$

Hence, g' and g are equivalent for r small.

In particular, $|\det g_{ij}(x, t) - \det g'_{ij}(x, t)| \leq ct$ for a positive c independent of x , p , and t . Since M has bounded geometry, $\det g_{ij}$ is uniformly bounded on all of M both from above and away from zero. Therefore the volume forms for g and g' are equivalent.

An estimate similar to (32) holds for $(g')^{ij}(x, t) - g^{ij}(x, t)$. Together with the relation $|\alpha|_g^2(p) = \sum_{i,j} g^{ij}(p) \alpha_p(e_i) \alpha_p(e_j)$ for a one-form α , this gives the claimed result for the one-forms \square

Remark 4.4. The proof implies that the constant c in the Poincaré inequality can be chosen to only depend on the bounds of R^M and its derivatives, on bounds of Π on r_{FC} as defined in Definition 4.1, on $p \in [1, \infty]$, and on the width.

We note that starting with $f(t) = f(0) + \int_0^t f'(s) ds$ in Section 4.1 the proof immediately generalizes and gives

Theorem 4.5. *Let M be an m -dimensional smooth Riemannian manifold with smooth boundary $\partial M = \partial_D M \sqcup \partial_N M$ and with bounded geometry. Let $E \rightarrow M$ be a vector bundle with bounded geometry. Assume that $(M, \partial_D M)$ has finite width, i.e. $M \subset U_R(\partial_D M)$, for some $R \in (0, \infty)$. For every $p \in [1, \infty]$ there exists $0 < c = c_{M,p} < \infty$ such that*

$$\|f\|_{L^p(M,E)} \leq c \left(\|f\|_{L^p(\partial_D M, E|_{\partial_D M})} + \|\nabla f\|_{L^p(M,E)} \right)$$

for all $f \in H^1(M; E)$.

4.3. Well-posedness in H^1 . The results of the previous three sections combine now to give the following. We assume in this subsection that $(M, \partial_D M)$ has finite width.

Corollary 4.6. *Let us assume that $(M, \partial_D M)$ has finite width and that $E \rightarrow M$ has bounded geometry. Then the semi-norm $|u|_{H^1(M;E)} := \|\nabla u\|_{L^2(M;E)}$ is a norm and it is equivalent to the H^1 -norm on $H_D^1(M; E)$.*

Proof. This follows directly from Lemma 2.11 since $(M, \partial_D M)$ satisfies the 2-Poincaré inequality, by Theorem 3.10 just proved. \square

Remark 4.7. We notice that, unlike the compact case, it we cannot replace $\partial_D M$ with a subset A of large measure, as is shown by the example $M = [0, 1] \times \mathbb{R}$ and $A = \{0\} \times [0, \infty)$, since functions vanishing on A do not satisfy a Poincaré inequality.

Theorem 2.12 and Corollary 2.13 then give right away the following result.

Theorem 4.8. *Assume that $(M, \partial_D M)$ has finite width, $E \rightarrow M$ has bounded geometry, and use the notation in Theorem 2.12. In particular, $P = P_a + Q$ is uniformly strongly elliptic.*

- (i) *If $c \in L^\infty(M; \text{End}(E))$, $\Re(c) \geq R_\eta$, $\eta > 0$, then $P + c: H_D^1(M) \rightarrow H_D^1(M)^*$ is an isomorphism.*
- (ii) *If $c \in L^\infty(M; \text{End}(E))$, $\Re(c) \geq 0$ and $Q + Q^*: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ of small enough norm, then $P + c: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism.*
- (iii) *$P_g: H_D^1(M; E) \rightarrow H_D^1(M; E)^*$ is an isomorphism.*

5. APPLICATIONS

We continue to assume that M is a smooth manifold with smooth boundary and bounded geometry. In this section, we record what is one of our main applications of the Poincaré inequality, that is, the well-posedness of the Dirichlet (or Poisson) problem on manifolds with finite width in *higher* Sobolev spaces.

5.1. Sobolev spaces, partitions of unity, and non-zero boundary conditions. In the following, we will need also a local description of the Sobolev spaces using partitions of unity for the manifold and synchronous trivializations for the vector bundle. To that end, we need suitable coverings of our manifold. For the sets in the covering that are away from the boundary, we will use geodesic normal coordinates, whereas for the sets that intersect the boundary, we will use Fermi coordinates as in Definition 4.1. This works well for manifolds with bounded geometry, so *we will assume in this subsection that M^m is a manifold with smooth boundary and that $E \rightarrow M$ is a hermitian vector bundle of bounded geometry.* Note that in this subsection, we *do not* assume that M has finite width.

Recall that if $p \in \partial M$, then $U_p(r) := \kappa_p(B_r^{m-1}(0) \times [0, r))$ is the image of the Fermi coordinate map κ_p of Equation (30). On the other hand, if $\text{dist}(p, \partial M) \geq r$, then recall that $U_p(r) := \exp_p^M(B_r^m(0)) = \{x \in M \mid \text{dist}(x, p) < r\}$. See Equation (31).

Definition 5.1. Let M^m be a manifold with boundary and bounded geometry and let $0 < r \leq r_{FC} := \min \{ \frac{1}{2} r_{\text{inj}}(\partial M), \frac{1}{4} r_{\text{inj}}(M), \frac{1}{2} r_{\partial} \}$, as in Definition 4.1. A subset $\{p_\gamma\}_{\gamma \in \mathbb{N}}$ is called an *r -covering subset of M* if the following conditions are satisfied:

- (i) For each $R > 0$, there exists $N_R \in \mathbb{N}$ such that, for each $p \in M$, the set $\{\gamma \in \mathbb{N} \mid \text{dist}(p_\gamma, p) < R\}$ has at most N_R elements.
- (ii) For each $n \in \mathbb{N}$, we have either $p_\gamma \in \partial M$ or $d(p_\gamma, \partial M) \geq r$, so that $U_\gamma := U_{p_\gamma}(r)$ is defined.
- (iii) $M \subset \cup_{\gamma=1}^\infty U_\gamma$.

Remark 5.2. It follows from the results in [34] that if $0 < r < r_{FC}$ then an r -covering subset of M always exists, since M is a manifold with boundary and bounded geometry. The picture below shows an example of an r -covering set, where, the p_β 's denote the points $p_\gamma \in \partial M$ and the p_α 's denote the rest of the points of $\{p_\gamma\}$.

Remark 5.3. Let M be a manifold with boundary and bounded geometry. Let $\{p_\gamma\}_{\gamma \in \mathbb{N}}$ be an r -covering set and $\{U_\gamma\}$ be the associated covering of M . It follows from (i) of the definition that the coverings $\{U_\gamma\}$ of M and $\{U_\gamma \cap \partial M\}$ of ∂M are *uniformly locally finite*, i.e. there is an $N_0 > 0$ such that no point belongs to more than N_0 of the sets U_γ .

We shall need the following class of partitions of unity defined using r -covering sets. Recall the definition of the sets U_γ from Equation (31).

Definition 5.4. A partition of unity $\{\phi_\gamma\}_{\gamma \in \mathbb{N}}$ of M is called an *r -uniform partition of unity associated to the r -covering set $\{p_\gamma\} \subset M$* (see Definition 5.1) if

- (i) The support of each ϕ_γ is contained in U_γ .

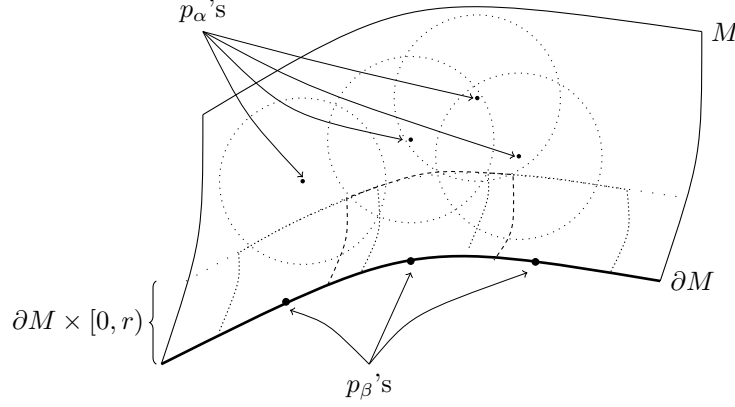


FIGURE 2. A uniformly locally finite cover by Fermi and geodesic coordinate charts, compare with Remark 5.2.

- (ii) For each multi-index α , there exists $C_\alpha > 0$ such that $|\partial^\alpha \phi_\gamma| \leq C_\alpha$ for all γ , where the derivatives ∂^α are computed in the normal geodesic coordinates, respectively Fermi coordinates, on U_γ .

Remark 5.5. Given an r -covering set S with $r \leq r_{FC}/4$, an r -uniform partition of unity associated to $S \subset M$ always exists, since M is a manifold with boundary and bounded geometry [34, Lemma 25].

Moreover, when working with a vector bundle $E \rightarrow M$, we need suitable trivializations adapted to the chosen Fermi coordinates. Since at the end it does not affect the proofs we are given here, we keep it short.

Definition 5.6. Let M be a manifold with boundary of bounded geometry, and let $E \rightarrow M$ be a hermitian vector bundle of bounded geometry. Let $(U_{p_\gamma}, \kappa_\gamma, \phi_\gamma)$ be Fermi coordinates on M together with an associated r -uniform partition of unity as in Definitions 4.1, 5.1 and 5.4. If $p_\gamma \in M \setminus M \setminus U_r(\partial M)$, then $E|_{U_{p_\gamma}}$ is trivialized by parallel transport along radial geodesics emanating from p_γ , see [34, Definition 5.2] for a more explicit description. If $p_\gamma \in \partial M$, then we trivialize $E|_{U_{p_\gamma}}$ as follows: First we trivialize $E|_{U_{p_\gamma}N}$ along the underlying geodesic coordinates on ∂M . Then, we trivialize by parallel transport along geodesics emanating from ∂M and being normal to ∂M , compare [34, Definition 5.12]. The resulting trivializations are called *synchronous trivializations along Fermi coordinates* and are maps

$$\xi_\gamma: \kappa_{p_\gamma}^{-1}(U_{p_\gamma}) \times \mathbb{C}^k \rightarrow E|_{U_{p_\gamma}}$$

where k is the rank of E .

We have then the following proposition that is a consequence of Theorems 14 and 26 in [34]. See also [3, 6, 45, 67, 66] for related results, in particular, for the use of the partitions of unity.

Proposition 5.7. *Let M be a manifold with boundary and bounded geometry. Let $\{\phi_\gamma\}$ be a uniform partition of unity associated to the r -covering set $\{p_\gamma\} \subset M$ and*

let $\kappa_\gamma = \kappa_{p_\gamma}$ be as in Equation 4.1. Then

$$|||u|||^2 := \sum_{\gamma} \|(\phi_\gamma u) \circ \kappa_\gamma\|_{H^s}^2$$

defines a norm equivalent to the usual norm on $H^s(M)$, $s \in \mathbb{R}$. Here $\|\cdot\|_{H^s}$ is the H^s norm on either \mathbb{R}^m or on the half-space \mathbb{R}_+^m . The result remains true if we include a coefficient vector bundle $E \rightarrow M$ with bounded geometry and use the synchronous trivializations of Definition 5.6 and define

$$|||u|||^2 := \sum_{\gamma} \|\xi_\gamma^*(\phi_\gamma u)\|_{H^s}^2$$

Similarly, we have the following extension of the trace theorem to the case of a manifold M with boundary and bounded geometry and a hermitian vector bundle $E \rightarrow M$ of bounded geometry, see Theorem 5.14 in [34]. (See also [6] for the case of Lie manifolds.)

Theorem 5.8 (Trace theorem). *Let M be a manifold with boundary and bounded geometry. Then, for every $s > 1/2$, the restriction to the boundary $\text{res}: \Gamma_c(M) \rightarrow \Gamma_c(\partial M)$ extends to a continuous, surjective map*

$$\text{res}: H^s(M) \rightarrow H^{s-\frac{1}{2}}(\partial M).$$

If $s = 1$, then the kernel of this map coincides with $W_0^{1,2}(M)$. The result remains true if we include a coefficient vector bundle $E \rightarrow M$ with bounded geometry.

Combining the trace theorem with Theorem 4.8, we obtain the following result. We use the notation of Theorems 2.12 and 4.8. The constant R_η is as in Lemma 2.6.

Theorem 5.9. *We use the notation in Theorem 4.8. In particular, $(M, \partial_D M)$ has finite width and $P = P_a + Q$ is uniformly strongly elliptic. Let $\tilde{P}: H^1(M; E) \rightarrow H_D^1(M; E)^* \oplus H^{1/2}(\partial_D M; E)$ be given by $\tilde{P}u = ((P + c)u, u|_{\partial_D M})$.*

- (i) *If $c \in L^\infty(M; \text{End}(E))$, $\Re(c) \geq R_\eta$ with R_η as in Lemma 2.6 for some $\eta \in (0, c_a)$ then \tilde{P} is an isomorphism.*
- (ii) *If $c \in L^\infty(M; \text{End}(E))$, $\Re(c) \geq 0$, and $\|Q + Q^*\|$ is small, then \tilde{P} is an isomorphism.*

If the assumption of this theorem are satisfied, we obtain the usual statement that

$$(33) \quad \begin{cases} (P_a + Q + c)u = F \in H_D^1(M; E)^* & \text{in } M \\ u = g \in H^{1/2}(\partial_D M; E) & \text{on } \partial_D M, \end{cases}$$

has a unique solution in $H^1(M; E)$ and this solution depends continuously on F and g . We also note that the assumptions of the theorem are satisfied if $Q = 0$ and $c = 0$. In particular,

Corollary 5.10. *Assume that $(M, \partial_D M)$ has finite width. Then $\tilde{\Delta}_g u = (\Delta_g u, u|_{\partial M})$ yields an isomorphism $\tilde{\Delta}_g: H^1(M) \rightarrow H_D^1(M)^* \oplus H^{1/2}(\partial_D M)$.*

5.2. Compactness of the family of local operators and higher regularity for pure Dirichlet boundary conditions. A partition of unity argument allows us to extend the well-posedness result of Corollary 2.13 to higher regularity Sobolev spaces. We present here a simple argument that works for smooth coefficient operators. To this end, we assume that a , the bounded family of sesquilinear forms on $T^*M \otimes_{\mathbb{R}} E$ and all the other coefficients of our operator $P := P_a$ are smooth. We continue to assume that a is bounded and P is strongly elliptic. (We drop a from the notation below.)

We shall use the notation introduced in Definition 5.1. We then denote by P_x the corresponding operator on the Euclidean ball, respectively cylinder, corresponding to the geodesic normal coordinates, respectively Fermi coordinates, on U_x . Thus $\langle P_x u, v \rangle = \int_{U_x} a(du, dv) \, \text{dvol}_g$.

Lemma 5.11. *Let us assume that all the covariant derivatives of a are bounded in the coordinates of Definition 5.1 for some $r < r_{FC}$. Then the set $\{P_x \mid \text{dist}(x, \partial M) \geq r\}$ is a relatively compact subset of the set of differential operators on the closed ball $\overline{B_r^m(0)} \subset \mathbb{R}^m$. Similarly, the set $\{P_y \mid y \in \partial M\}$ is a relatively compact subset of the set of second differential operators on $b(r) := B_r^{m-1}(0) \times [0, r] \subset \mathbb{R}^m$.*

Proof. The coefficients of the operators P_x and all their derivatives are uniformly bounded, by assumption. The Arzela-Ascoli theorem then yields the result. \square

We get the following proposition

Proposition 5.12. *Under the assumptions of Lemma 5.11, if P is also uniformly strongly elliptic, then for any $k \in \mathbb{N}$ (so $k \geq 1$), there exists $C > 0$ such that*

$$\|w\|_{H^{k+1}(M;E)} \leq C \left(\|Pw\|_{H^{k-1}(M;E)} + \|w\|_{H^k(M;E)} \right),$$

for any $w \in H_0^1(U_\gamma; E|_{U_\gamma})$, where U_γ is a coordinate patch of Definition 5.1.

Proof. We start with the case of E being a trivial bundle and explain the small adjustments to general E at the end. Elliptic regularity for strongly elliptic equations (see, for instance, Theorem 8.13 in [33], or Theorem 9.3.3 in [40]; see also Proposition 11.10 in [63]) gives, for every $\gamma \in I$ as in Definition 5.1, that there exists C_γ such that

$$\|w\|_{H^{k+1}(M)} \leq C_\gamma \left(\|Pw\|_{H^{k-1}(M)} + \|w\|_{H^k(M)} \right)$$

for any $w \in H_0^1(U_\gamma)$. This should be understood in the sense that if the right hand side is finite, then $\|w\|_{H^{k+1}(M)} < \infty$, and hence $w \in H^{k+1}(M)$. We need to show that we can choose C_γ independent of γ . Let us assume the contrary. Then, for a suitable subsequence p_j , we have that there exist $w_j \in H_0^1(U_j)$ such that

$$\|w_j\|_{H^{k+1}(M)} > 2^j \left(\|Pw_j\|_{H^{k-1}(M)} + \|w_j\|_{H^k(M)} \right).$$

We can assume that the points p_j are either all at distance at least r to the boundary, or that they are all on the boundary. Let us assume that they are all on the boundary. The other case is very similar (even simpler). Using Fermi coordinates $\kappa_j: b(r) = B_r^{m-1}(0) \times [0, r] \rightarrow U_j$ we can move to a fixed cylinder, but with the price of replacing P with P_j . By Lemma 5.11, after passing to another subsequence, we can assume that the coefficients of the corresponding operators P_j converge in the \mathcal{C}^∞ topology on $b(r)$ to the coefficients of a fixed operator P_∞ . In particular, we can assume that

$$(34) \quad \|P_j w\|_{H^{k-1}(b(r))} \geq \|P_\infty w\|_{H^{k-1}(b(r))} - \epsilon_j \|w\|_{H^{k+1}(b(r))},$$

for all $w \in H^{k+1}(b(r))$, where $\epsilon_j \rightarrow 0$ as $j \rightarrow \infty$ independent of w . Then P_∞ is still strongly elliptic, since the condition of Equation (15) is preserved by uniform limits (and the parameter c_a does not depend on j). This gives that there exists $C > 0$ independent of j such that $C(\|P_\infty w\|_{H^{k-1}(b(r))} + \|w\|_{H^k(b(r))}) \geq \|w\|_{H^{k+1}(b(r))}$ for all $w \in H_0^{k+1}(b(r))$.

By Proposition 5.7 we also have

$$(35) \quad \frac{1}{1+C_0} \leq \frac{\|w \circ \kappa_j^{-1}\|_{H^\ell(U_j)}}{\|w\|_{H^\ell(b(r))}} \leq 1 + C_0,$$

for some $C_0 > 0$, for all $w \in H^{k+1}(b(r))$, and for $\ell \leq k+1$, by the bounded geometry of M .

Equations (34) and (35) then give

$$\begin{aligned} \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} &\leq C(\|P_\infty(w_j \circ \kappa_j)\|_{H^{k-1}(b(r))} + \|w_j \circ \kappa_j\|_{H^k(b(r))}) \\ &\leq C(\|P_j(w_j \circ \kappa_j)\|_{H^{k-1}(b(r))} + \|w_j \circ \kappa_j\|_{H^k(b(r))} \\ &\quad + \epsilon_j \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))}) \\ &\leq C(1+C_0)(\|P w_j\|_{H^{k-1}(M)} + \|w_j\|_{H^k(M)}) \\ &\quad + C\epsilon_j \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \\ &\leq 2^{-j}C(1+C_0)\|w_j\|_{H^{k+1}(M)} + C\epsilon_j \|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \\ &\leq C((1+C_0)2^{-j} + \epsilon_j)\|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \end{aligned}$$

and this is a contradiction for large j , since $\|w_j \circ \kappa_j\|_{H^{k+1}(b(r))} \neq 0$. Let now E be a general hermitian vector bundle. Then the proof above remains the same, only the notation slightly varies since one needs to replace $w_j \circ \kappa_j$ by $\xi_j^* w_j$ where ξ_j is the synchronous trivialization along the chosen Fermi coordinates, compare Definition 5.6. \square

Results of this type can be used to show that $-\Delta_g$ generates an analytic semi-group on $L^2(M)$ (see [53], for example). In the same spirit, it shows that the domain of Δ_g^k is contained in H^{2k} .

Combining with Theorem 5.9 and Corollary 5.10, we immediately obtain the following results. We formulate them only for Dirichlet boundary conditions, since considering Neumann boundary conditions would take us too far from the initial goals. To stress the analogy with our previous results, we notice that $H^{\ell-1}(M; E) \subset H^{-1}(M; E) := H_0^1(M; E)^*$.

Theorem 5.13. *We use the notation in Theorem 4.8. In particular, $(M, \partial M)$ has finite width and $P = P_a + Q$ is uniformly strongly elliptic. Assume that all the coefficients of P are in $W^{\infty, \infty}$. Let $\tilde{P}: H^{\ell+1}(M; E) \rightarrow H^{\ell-1}(M; E) \oplus H^{\ell+1/2}(\partial M; E|_{\partial M})$ be given by $\tilde{P}u = ((P+c)u, u|_{\partial M})$.*

- (i) *If $c \in W^{\ell, \infty}(M; \text{End}(E))$, $\Re(c) \geq R_\eta$ with R_η as in Lemma 2.6 for some $\eta \in (0, c_a)$, then \tilde{P} is an isomorphism.*
- (ii) *If $c \in W^{\ell, \infty}(M; \text{End}(E))$, $\Re(c) \geq 0$, and $Q + Q^*: H_0^1(M; E) \rightarrow H^{-1}(M; E)$ is small, then \tilde{P} is an isomorphism.*

In particular,

Corollary 5.14. *Assume that $(M, \partial M)$ has finite width. Then $\tilde{\Delta}_g u = (\Delta_g u, u|_{\partial M})$ yields isomorphisms $\tilde{\Delta}_g: H^{\ell+1}(M) \rightarrow H^{\ell-1}(M) \oplus H^{\ell+1/2}(\partial M)$, $\ell \in \mathbb{Z}_+$.*

We have the following standard form of well-posedness for boundary value problems.

Corollary 5.15. *Assume $(M, \partial M)$ has finite width, $E \rightarrow M$ is of bounded geometry, and $\ell \in \mathbb{Z}_+$. Assume also that $P = P_a + Q$ and c satisfy the assumptions of Theorem 5.13 (this includes the case $Q = 0$ and $c = 0$). Then the problem*

$$(36) \quad \begin{cases} (P_a + Q + c)u = F \in H^{\ell-1}(M; E) & \text{in } M \\ u = g \in H^{\ell+1/2}(\partial M; E) & \text{on } \partial_D M, \end{cases}$$

has a unique solution $u \in H^{\ell+1}(M; E)$ and this solution depends continuously on F and g .

5.3. Extensions. A first possible extension is to mixed boundary conditions (Dirichlet on $\partial_D M$ and Neumann on $\partial_N M$). To this end, one considers

$$(37) \quad F(u) := \int_M f u \, d\text{vol}_g + \int_{\partial_N M} u h dS,$$

where dS is the induced volume form on the boundary. One would have to replace the operator \tilde{P} of Theorem 5.13 with

$$(38) \quad \begin{aligned} \tilde{P}: H^{\ell+1}(M; E) &\rightarrow H^{\ell-1}(M; E) \oplus H^{\ell+1/2}(\partial_D M; E) \oplus H^{\ell-1/2}(\partial_N M; E) \\ \tilde{P}(u) &:= (Pu, u|_{\partial_D M}, \partial_\nu^\alpha u), \end{aligned}$$

where ∂_ν^α is the conormal derivative of u at the boundary, given by $\sum a_{ij} \nu_i \partial_j$ in local coordinates, where P has principal part $\sum_{ij} a_{ij} \partial_i \partial_j$. For instance, Corollary 5.15 becomes

Corollary 5.16. *Assume that $(M, \partial_D M)$ has finite width and that $\ell \in \mathbb{Z}_+$. Assume $P = P_a + Q$ and c satisfy the assumptions of Theorem 5.13. Then the problem*

$$(39) \quad \begin{cases} (P_a + Q + c)u = F \in H^{\ell-1}(M; E) & \text{in } M \\ u = h_0 \in H^{\ell+1/2}(\partial_D M; E) & \text{on } \partial_D M \\ \partial_\nu u = h_1 \in H^{\ell-1/2}(\partial_N M; E) & \text{on } \partial_N M, \end{cases}$$

has a unique solution $u \in H^{\ell+1}(M; E)$ continuously depending on F , h_0 , and h_1 .

The proofs of the results in the case of mixed boundary conditions is essentially the same as in the case of pure Neumann boundary conditions, but notice that the condition that $(M, \partial_D M)$ be of finite width is more restrictive than the condition that $(M, \partial M)$ be of finite width. The main difference would be that one would have to properly formulate and extend Proposition 5.12 to obtain regularity of Neumann boundary conditions. To this end, one would need that uniformly strongly elliptic operators with Neumann boundary conditions satisfy the higher regularity (or Shapiro-Lopatinski) condition. This is much less discussed in the literature than the case of Dirichlet boundary conditions. See however [63]. The results and proofs of that book can be adapted to give the straightforward, but long and tedious proof of the regularity result that we need for Neumann boundary conditions.

A further extension of the results of the previous subsections is to use a conformal change of coordinates to obtain a well-posedness result on polygonal or cuspidal

domains in two dimensions, provided that no two adjacent edges have Neumann boundary conditions (this is in order to ensure that the resulting pair $(M, \partial_D M)$ has finite width). This would recover, in particular, a celebrated result of Kondratiev [44] (see also [25, 46]). To this end, one would use a conformal invariance of the Laplacian in two dimensions. A different proof can also be extended to higher dimensions [19]. Let Ω be a bounded domain with conical points and r denote the distance to the boundary. Let $\mathcal{K}_a^\ell(\Omega) := \{u \mid r^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), |\alpha| \leq m\}$. Here is the statement of Kondratiev's result.

Theorem 5.17 (Kondratiev). *Let Ω be a bounded domain with conical points. There exists $\eta > 0$ such that the Laplacian induces an isomorphism*

$$\Delta: \mathcal{K}_{a+1}^{\ell+1}(\Omega) \cap \{u \mid \partial_D u = 0\} \cap \{\partial_n u \mid \partial_N \Omega = 0\} \rightarrow \mathcal{K}_{a-1}^{\ell-1}(\Omega)$$

for $|a| < \eta$, provided that no two adjacent edges have Neumann boundary conditions.

The applications to boundary value problems in this paper are based on “energy methods.” An important alternative method is to use integral kernel methods (the method of “layer potentials”) that would require pseudodifferential operators. Related results were obtained in [15, 16, 21, 50, 59]. See also [48, 69] for applications of regularity results of this kind to degenerate boundary value problems. We hope our results will shed light also on integral kernel methods.

6. CHARACTERIZATION OF MANIFOLDS WITH BOUNDED GEOMETRY

The goal of this subsection is to provide a criterion for a manifold with boundary to be of bounded geometry.

Theorem 6.1. *Assume that (M, g) is a Riemannian manifold with smooth boundary such that:*

(N) *There is $r_\partial > 0$ such that*

$$\partial M \times [0, r_\partial] \rightarrow M, (x, t) \mapsto \exp^\perp(x, t) := \exp_x(t\nu_x)$$

is a diffeomorphism onto its image.

(I) *There is $r_{\text{inj}}(M) > 0$ such that for all $r \leq r_{\text{inj}}(M)$ and all $x \in M \setminus U_r(\partial M)$, the exponential map $\exp_x: B_r^{T_x M}(0) \subset T_x M \rightarrow M$ defines a diffeomorphism onto its image.*

(B) *For every $k \geq 0$, we have*

$$\|\nabla^k R\|_{L^\infty} < \infty \quad \text{and} \quad \|(\nabla^{\partial M})^k \Pi\|_{L^\infty} < \infty.$$

Then (M, g) is a Riemannian manifold with boundary and bounded geometry in the sense of Definition 3.4.

Remark 6.2. The theorem implies in particular, that our definition of a manifold with boundary and bounded geometry coincides with the one given by Schick in [58, Definition 2.2]. According to Schick's definition a manifold with boundary has bounded geometry if it satisfies (N), (I) and (B) and if the boundary is itself has positive injectivity radius. One of the statements in the theorem is that (N), (I) and (B) imply that the boundary has bounded geometry.

Remark 6.3. In [17] Botvinnik and Müller defined manifolds with boundary with (c, k) -bounded geometry. Their definition differs from our definition of manifolds with boundary and bounded geometry in several aspects, in particular they only control k derivatives of the curvature.

Remark 6.4. For later use, we provide several of the auxiliary lemmata needed to prove Theorem 6.1 not just for the boundary of a manifold, but also for submanifolds.

6.1. Preliminaries on the injectivity radius. Let (M, g) be a Riemannian manifold without boundary and $p \in M$. We write $r_{\text{inj}}(p)$ for the injectivity radius of (M, g) in p (that is the supremum of all r such that $\exp: B_r^{T_p M}(0) \subset T_p M \rightarrow M$ is injective, as before).

We define the *curvature radius* of (M, g) in p as

$$\rho := \sup\{r > 0 \mid \exp_p \text{ is defined on } B_{\pi r}^{T_p M}(0) \text{ and } |\sec| \leq 1/r^2 \text{ on } B_{\pi r}^M(p)\},$$

where \sec is the sectional curvature and $|\sec| \leq 1/r^2$ is the short notation for $|\sec(E)| \leq 1/r^2$ for all planes $E \subset T_q M$ and all $q \in B_{\pi r}^M(p)$.

Standard estimates in Riemannian geometry, see e.g. [27], show that the exponential map is an immersion on $B_{\pi\rho}^M(p)$. In other words, no point in $B_{\pi\rho}^M(p)$ is conjugate to p with respect to geodesics of length $< \pi\rho$.

If there is a geodesic γ from p to p of length $2\delta > 0$, then it is obvious that $r_{\text{inj}}(p) \leq \delta$. The converse is true if δ is small:

Lemma 6.5 ([56, Prop. III.4.13]). *If $r_{\text{inj}}(p) < \pi\rho$, then there is geodesic of length $2r_{\text{inj}}(p)$ from p to p .*

We will use the following theorem, due to Cheeger, see [22]. We also refer to [54, Sec. 10, Lemma 4.5] for an alternative proof, which easily generalizes to the version below.

Theorem 6.6 (Cheeger). *Let (M, g) be a Riemannian manifold with $|R| \leq K$. Let $A \subset M$. We assume that there is an $r_0 > 0$ such that \exp_p is defined on $B_{r_0}^{T_p M}(0)$ for all $p \in A$. Let $\rho > 0$. Then $\inf_{p \in A} r_{\text{inj}}(p) > 0$ if, and only if, $\inf_{p \in A} \text{vol}(B_\rho^M(p)) > 0$.*

6.2. Controlled submanifolds are of bounded geometry.

Definition 6.7. Let M be a Riemannian manifold with boundary. We say that the boundary is *controlled* if $\|(\nabla^{\partial M})^k \Pi\|_{L^\infty} < \infty$ for all k , and if (N) from above is satisfied.

Let N^n be a submanifold of a Riemannian manifold M^m . Let $\nu_N \subset TM|_N$ be the normal bundle of N in M . The second fundamental form of N is defined as the map $\Pi: TN \times TN \rightarrow \nu_N$ with $\Pi(X, Y) := \nabla_X Y - \nabla_X^N Y$. For $X \in T_p N$ and $s \in \Gamma(\nu_N)$ one can decompose $\nabla_X s$ in the $T_p N$ component (which is given by the second fundamental form) and the normal component $\nabla_X^\perp s$. This ∇^\perp yields a connection on the bundle $\nu_N \rightarrow N$. We write R^\perp for the associated curvature. The normal exponential map \exp^\perp is the restriction of $\exp^M: \mathcal{D} \subset TM \rightarrow M$ to $\mathcal{D}^\perp := \mathcal{D} \cap \nu_N$, where \mathcal{D} is the maximal domain of definition of the exponential map.

Definition 6.8. Let M^m be a Riemannian manifold without boundary. We say that a closed submanifold $N \subset M$ is *controlled* if $\|(\nabla^N)^k \Pi\|_{L^\infty} < \infty$ for all $k \geq 0$ and there is $r_\partial > 0$ such that

$$\exp^\perp: U_{r_\partial}(\nu_N) \rightarrow M$$

is injective where $U_r(\nu_N)$ is the set of all vectors in ν_N of length $< r$.

We will show that the geometry of controlled submanifolds is 'bounded'. Thus, later we will rename those as *bounded geometry submanifolds*.

Lemma 6.9. *For every Riemannian manifold (M, g) with boundary and bounded geometry there is a complete Riemannian manifold \widehat{M} without boundary such that $\|\nabla^k R\|_{L^\infty} < \infty$ for all $k \geq 0$, that $M \rightarrow \widehat{M}$ is an isometric embedding, and that ∂M is a controlled submanifold of \widehat{M} .*

Proof. The metric $(\exp^\perp)^*g$ on $\partial M \times [0, r_\partial)$ is of the form $(\exp^\perp)^*g = h_r + dr^2$ where $r \in [0, r_\partial)$ and where h_r , $r \in [0, r_\partial)$, is a family of metrics on ∂M such that $(\partial/\partial r)^k h_r$ is a bounded tensor for any $k \in \mathbb{N}$. Using a cut-off argument it is possible to define h_r also for $r \in (-1 - r_\partial, 0)$ such all $(\partial/\partial r)^k h_r$ are bounded tensors, and such that $h_r = h_{-1-r}$ for all $r \in (-1 - r_\partial, r_\partial)$. An immediate calculation shows that then $(\partial M \times (-1 - r_\partial, r_\partial), h_r + dr^2)$ has bounded curvature R , and all derivatives of R are bounded. We now obtain \widehat{M} by gluing together two copies of M together with $\partial M \times (-1 - r_\partial, r_\partial)$. Obviously the curvature and all its derivatives are bounded on \widehat{M} . \square

Lemma 6.10. *Let M^m be a complete Riemannian manifold with boundary ∂M satisfying (N), (I), and (B) as in Theorem 6.1. Then the injectivity radius of \widehat{M} as constructed in the last lemma is positive.*

Proof. We apply (I) for $r := \frac{1}{2} \min\{r_{\text{inj}}(M), r_\partial\}$. Then $\inf_{q \in M \setminus U_r(\partial M)} r_{\text{inj}}(q) > 0$. It thus remains to show $\inf_{q \in \partial M \times [-1-r, r]} r_{\text{inj}}(q) > 0$. Let $q = (x, t) \in \partial M \times [-1 - r, r]$. We define the diffeomorphism $f_q: \partial M \times (t - r, t + r) \rightarrow \partial M \times (0, 2r) \subset M$, $(y, s) \mapsto (y, s - t + r)$. Then the operator norms $\|(df_q)\|$ and $\|(df_q)^{-1}\|$ are bounded by $C_1 \geq 1$ that only depends on (M, g) and the chosen extension \widehat{M} , but not on q . Theorem 6.6 for $\rho = r/C_1$ gives $v > 0$ such that $\text{vol}(B_{r/C_1}^M(z)) > v$ for all $z \in M \setminus U_r(\partial M)$. Together with $B_{r/C_1}^M(f_q(q)) \subset f_q(B_r^{\widehat{M}}(q)) \subset \partial M \times (0, 2r)$ we get $\text{vol}(B_r^{\widehat{M}}(q)) \geq C_1^{-m} \text{vol}(f_q(B_{r/C_1}^M(q))) \geq C_1^{-m} \text{vol}(B_{r/C_1}^M(q)) \geq C_1^{-m} v$. Using again Theorem 6.6 for $\rho = r$ we obtain the required statement. \square

Lemma 6.11. *Let M be a Riemannian manifold without boundary and with totally bounded curvature (i.e. $\|\nabla^k R^M\|_{L^\infty} < \infty$, for all $k \geq 0$). Let $N \subset M$ be a submanifold with $\|(\nabla^N)^k \Pi\|_{L^\infty} < \infty$ for all $k \geq 0$. Then*

- (i) $\|(\nabla^N)^k R^N\|_{L^\infty} < \infty$ for all $k \geq 0$.
- (ii) *The curvature of the normal bundle of N in M and all its covariant derivatives are bounded. In other words $\|(\nabla^N)^k R^\perp\|_{L^\infty} \leq c_k$ for all $k \in \mathbb{N}_0$.*

Proof. (i) This is [34, Lem. 22].

(ii) For a normal vector field η let W_η be the Weingarten map for η , i.e. for $X \in T_p N$ let $W_\eta(X)$ be the tangential part of $-\nabla_X \eta$. Thus $\langle \Pi(X, Y), \eta \rangle = \langle W_\eta(X), Y \rangle$ for the vector-valued second fundamental form Π . The Ricci equation [14, p. 5] equation states that the curvature of the normal bundle is given by the following formula

$$(40) \quad g^M(R^\perp(X, Y)\eta, \zeta) = g^M(R(X, Y)\eta, \zeta) - g^M(W_\eta(X), W_\zeta(Y)) + g^M(W_\eta(Y), W_\zeta(X))$$

where $X, Y \in T_p N$, and where $\eta, \zeta \in T_p M$ are normal to N . The boundedness of $R(X, Y)$ and Π thus implies the boundedness of R^\perp . For bounding $(\nabla^N)^k R^\perp$

one has to derive 40 covariantly, and the difference between ∇ and ∇^N provides additional terms linear in Π . Thus, one iteratively sees that $(\nabla^N)^k R^\perp$ is a polynomial in the variables Π , $\nabla^N \Pi$, \dots , $(\nabla^N)^k \Pi$, R , ∇R , \dots , $\nabla^k R$, and thus bounded. \square

The total space of the normal bundle ν_N carries a natural Riemannian metric: The connection ∇^\perp defines a splitting of the tangent bundle of this total space into the vertical tangent space (which is the kernel of the differential of the foot point map) and the horizontal tangent space (given by the connection). The horizontal space inherits the Riemannian metric of N and the vertical tangent spaces are canonically isomorphic to the fibers of the normal bundle and they carry the canonical metric. Let $\pi: \nu_N \rightarrow N$ be the foot point map. Then π is a Riemannian submersion.

It is clear from the definition of the normal exponential map that the bounds on $\nabla^k R$ and on $(\nabla^N)^k \Pi$ imply that there is an $r_1 \in (0, r_\partial)$ such that the derivative dF and its inverse $(dF)^{-1}$, for $F := \exp^\perp|_{U_{r_1}(\nu_N)}$, are uniformly bounded—say $\|dF\| \leq C_2$ and $\|(dF)^{-1}\| \leq C_2$ with $C_2 \geq 1$.

Lemma 6.12. *Let M^m be a Riemannian manifold without boundary and of bounded geometry. Let N^n be a controlled submanifold of M . Then the injectivity radius of N is positive.*

Proof. The proof is similar to the one of Lemma 6.10. Let $q \in N$. We consider $B(q) := U_{r_1}(\nu_N) \cap \pi^{-1}(B_r^N(q))$. Then, $B_{r_1/C_2}^M(q) \subset F(B(q))$ by boundedness of $(dF)^{-1}$. By Theorem 6.6 for M and $\rho = r_1/C_2$ there is a $v > 0$, independent of q , such that $\text{vol}(B_{r_1/C_2}^M(q)) \geq v$ and, thus, $\text{vol}(F(B(q))) \geq v$. Thus, $\text{vol}(B(q)) \geq C_2^{-n}v$ where the volume on $B(q)$ is taken w.r.t. the natural metric on ν_N . As π is a Riemannian submersion, $\text{vol}(B(q)) = r_1^{m-n} \text{vol}(B_{r_1}^N(q))$. Thus, $\text{vol}(B_{r_1}^N(q))$ is bounded from below independent on $q \in N$. The Theorem 6.6 applied for $\rho = r_1$ yields positive injectivity radius for N . \square

Corollary 6.13. *Let N be a controlled submanifold in a Riemannian manifold of bounded geometry. Then, N is a manifold of bounded geometry.*

Corollary 6.14. *Let M be a Riemannian manifold with boundary satisfying (N), (I) and (B). Then, ∂M is a manifold of bounded geometry.*

The last corollary together with Lemma 6.9 implies Theorem 6.1.

Remark 6.15. As we have seen that controlled submanifolds have bounded geometry in the sense of Theorem 6.1, we will call them bounded geometry submanifolds in further publications. In particular a hypersurface with unit normal fields are controlled if, only if, it is a bounded geometry hypersurface as in Definition 3.3.

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